Witness Sets, Polarity Reversal and the Processing of Quantified Sentences

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Experimental results show that the monotonicity of a quantifier \((Q)\) affects how it is processed [4, 7, 6]. \(Q\)'s are upward entailing (UE) if they permit inferences to supersets, i.e. from \(Q \, A \, B\) follows \(Q \, A \, B'\) for any \(B \subseteq B'\) (e.g. every, at least five). If from \(Q \, A \, B\) it follows that \(Q \, A \, B'\) for any \(B' \subseteq B\), the quantifier \(Q\) is downward entailing (DE) (e.g. no, at most five). As compared to UE \(Q\)'s, DE \(Q\)'s are more difficult to verify [4, 7] and draw inferences from [6]. Recent attempts at deriving predictions about the processing of \(Q\)'s are built on the notion of semantic automata [12], and have been presented e.g. in [11] and [10]. E.g., [11] improve on results by [8] by showing not only that the computational distinction between \(Q\)'s recognized by finite-automata and push-down automata is psychologically relevant, but also that there are differences in the time required for verifying statements involving \(Q\)'s even among the class of quantifiers recognized by finite state automata. However, since these approaches employ essentially the same kind of semantic automata for both DE and UE \(Q\)'s, they cannot explain why DE \(Q\)'s are more difficult to process than UE \(Q\)'s. To explain this, we formulate a quantification theory which predicts that the (expansion) operation employed in processing DE \(Q\)'s is more difficult than that for processing UE \(Q\)'s, because it involves (i) inferences from negative information and (ii) polarity reversal. The predictions of our account were tested in two experiments investigating the online comprehension and verification of simply (Exp. 1) and doubly quantified sentences (Exp. 2) with UE vs DE \(Q\)'s.

1 An algorithmic theory of quantification

In the following, we aim to provide an algorithmic theory of meaning in the sense of [9]: The meaning of an expression is the algorithm which computes its denotation. Here is the basic idea of what will be worked out below. In the case of a quantificational sentence \(Q_1 \, \text{boys} \, \text{tickled} \, Q_2 \, \text{girls}\), with \(Q_1 \, \text{boys}\) scoping over \(Q_2 \, \text{girls}\), the interpretation consists in specifying an algorithm which outputs whether S is true given any model, and the verification consists in applying this algorithm to a specific model.

Given a binary relation \(R\) (e.g. tickle), we first construct a polarity relation \(R^*\), such that \(\langle a, b, +\rangle \in R^*\) if \(\langle a, b \rangle \in R\); and \(\langle a, b, -\rangle \in R^*\) otherwise. We now specify an algorithm which derives \(\langle Q_1 \, \text{boys}, Q_2 \, \text{girls}, +\rangle\) if and only if our input sentence is true (relative to a certain scoping). The algorithm for the verification of the doubly quantified sentence above (relative to the subject wide scope reading) consists of the following steps:

(1) a. For every boy \(x\) who tickles some girl, determine which girls he tickles. If the set of girls tickled by \(x\) is a witness set of \(Q_2 \, \text{girls}\), add the positive information \(\langle x, Q_2 \, \text{girls}, +\rangle\). Otherwise, add the negative information that \(\langle x, Q_2 \, \text{girls}, -\rangle\).
   b. For every boy \(x\) who tickles no girl, add \(\langle x, Q_2 \, \text{girls}, +\rangle\) if \(Q_2 \, \text{girls}\) has the empty set as a witness set. Otherwise, add the negative information \(\langle x, Q_2 \, \text{girls}, -\rangle\).
Q₁ boys tickled Q₂ girls is true iff ⟨Q₁ boys, Q₂ girls, +⟩ can be added. Importantly, if neither Q₁ boys nor Q₂ girls have the empty set as a witness set, then the general algorithm can be simplified as follows:

(3) For every boy x who tickles some girl, determine which girls he tickles. If the set of girls tickled by x is a witness set of Q₂ girls, add the positive information ⟨x, Q₂ girls, +⟩.

(4) If for some boy x we have ⟨x, Q₂ girls, +⟩, and the set of boys y with ⟨y, Q₂ girls, +⟩ is a witness set of Q₁ boys then add ⟨Q₁ boys, Q₂ girls, +⟩. Otherwise, add ⟨Q₁ boys, Q₂ girls, −⟩.

In other words, if neither quantifier is an empty-set quantifier, the negative information can safely be ignored in every model, and the algorithm can be restricted to positive information. We will refer to this simplified procedure as simple expansion (s-exp). If one (or both) quantifiers are empty-set quantifiers, and the antecedent of (2b) holds, then negative information becomes relevant, since step (2b) allows the addition of positive information based on negative information, and therefore the more complex algorithm (c-exp) is required. The differentiation between s-exp and c-exp sets our theory apart from existing algorithmic proposals of quantification, as for instance [10]’s automata theory building on [12].

2 Quantification theory

First, we assume that simple NL determiners denote unary functions from restrictor sets to pairs consisting of the restrictor set itself and a set of subsets of the restrictor set, called the set of witness sets, i.e. as functions q : P(E) → P(E) × P(P(E)).

Definition 1 (w-function, w-quantifier, witness sets): Let q be a function from the set P(E) of subsets of the domain of entities E into P(E) × P(P(E)). Then q is called a w-function¹ if for any A ⊆ E there is a W ⊆ P(A), such that q(A) = ⟨A, W⟩. If q is a w-function and A ⊆ E, then q(A) = ⟨A, W⟩ is called a w-quantifier, and W the set of witness sets of q at A. If a w-quantifier q(A) = ⟨A, W⟩ is such that ∅ ∈ W, it is called an empty-set quantifier.

To illustrate, for any subset A of the domain of individuals E, we have:

(5) \[ \text{some}[A] = \langle A, \{ X : X \subseteq A \land |X| \geq 1 \} \rangle \]
(6) \[ \text{most}[A] = \langle A, \{ X : X \subseteq A \land |A \cap X| > |A - X| \} \rangle \]

Note that these w-quantifiers naturally correspond to the standard generalized quantifier denotations:

(6) \[ \text{some}_E[A] = \{ B : B \subseteq E \land |A \cap B| \geq 1 \} \]
(7) \[ \text{most}_E[A] = \{ B : B \subseteq E \land |A \cap (A \cap B)] |> |A - (A \cap B)| \} \]

Importantly, note that for most type ⟨1, 1⟩ quantifiers there is no corresponding w-quantifier:

\[ \text{EQUI}_E[A] = \{ B : B \subseteq E \land |A| = |B| \} \]
\[ \text{TOTAL}_E[A] = \{ B : B \subseteq E \land E = A \} \]

¹ Mnemonic for witness-set function.
In fact, a type \((1,1)\) quantifier \(Q\) has a corresponding w-quantifier if and only if \(Q\) satisfies both conservativity (CONS) and extension (EXT).\(^2\)

In order to explicitly encode positive and negative information we define the polarity relation \(P^*\) of an \(n\)-ary predicate: if \((a,b) \in P\) then \((a,b,+)) \in P^*, and if \((a,b) \notin P\) then \((a,b,-) \in P^*.\)

**Definition 2 (n-ary predicate, polarity relation):** A subset \(P\) of \((E \cup P(\mathcal{E})) \times P(\mathcal{P}(\mathcal{E})))^n,\) with \(n \geq 1\), is called an \(n\)-ary predicate. For any \(n\)-ary predicate \(P\) and \(\pi_i(\sigma)\) the \(i\)-th element of an \(n\)-ary tuple \(\sigma\) (for any \(1 \leq i \leq n\)), let \(P^* = \{\{\pi_1(\sigma), \ldots, \pi_n(\sigma), +\} : \sigma \in P\} \cup \{\pi_i(\sigma), \ldots, \pi_n(\sigma), -\} : \sigma \notin P\} be the polarity relation of \(P\).

To illustrate, if \(E = \{a,b,c\}, A = \{a,b\}, P = \{(a,a), (a,b), (c,c)\}\) is a binary predicate, then \(P^* = \{(a,a,+), (a,b,+), (c,c,+)\} \cup \{(a,c,-), (b,a,-), (b,b,-), (c,a,-), (c,b,-)\}\) is polarity relation of \(P\).

Next, we define the set of i-fillers \([\sigma]_i^{P^*}\) of a tuple \(\sigma \in P^*\) as the set of elements in the \(i\)-th position of an \(i\)-variant \(\tau \in P^*\) of \(\sigma\) (\(\tau\) is an \(i\)-variant of \(\sigma\) if \(\tau\) and \(\sigma\) differ at most at the \(i\)-th position).

**Definition 3 (i-fillers):** Let \(P^*\) be the polarity relation of some \(n\)-ary predicate \(P\) and let \(\sigma \sim_i \tau\) hold iff \(\sigma\) and \(\tau\) differ at most at the \(i\)-th element. For every \(\sigma \in P^*\) and any integer \(1 \leq i \leq n\), let \([\sigma]_i^{P^*} = \{\pi_i(\tau) : \tau \in P^* \land \tau \sim_i \sigma\}\) be the i-fillers of \(\sigma\) in \(P^*\). Let \([\sigma]_i^{P^*A} = [\sigma]_i^{P^*} \cap A\) be the set of those i-fillers of \(\sigma\) in \(P^*\) which are also in the set \(A\) (called restricter set).

For example, if \(P^* = \{(a,a,/+), (a,b,+), (b,a,-), (b,b,-)\}\), then \([a,a,+]^{P^*} = \{a\}, ([a,a,+])^{P^*} = \{a\}, ([b,a,-])^{P^*} = \{b\}, ([b,b,-])^{P^*} = \{b\}\). Finally, we formulate two expansion operations which, when applied to a quantifier \(q(A)\) and a polarity relation \(P^*\), add those tuples representing negative or positive information involving \(q(A)\) at position \(i\) in \(P^*\).

**Definition 4 (simple and complex i-expansion):** Let \(q : \mathcal{P}(\mathcal{E}) \rightarrow \mathcal{P}(\mathcal{E}) \times \mathcal{P}(\mathcal{P}(\mathcal{E}))\) be a function such that \(q(A) = \langle A, W \rangle\), where \(W \subseteq P(A)\), and \(P\) be an \(n\)-ary predicate over \(E\) \((n \geq 1)\). For any \(\sigma \in P^*\) and any \(1 \leq i \leq n\), let \([\sigma]_i^{P^*A}\) be the result of replacing the \(i\)-th element of \(\sigma\) by \(x\). Then the simple expansion of \(P^*\) by \(q(A)\) at position \(i\), written as \(s-exp\(_i\)(q(A), P^*)\), is the smallest set \(Q\) such that \(P^* \subseteq Q\) and clause 1 holds. The complex expansion of \(P^*\) by \(q(A)\) at position \(i\), written as \(s-exp\(_i\)(q(A), P^*)\), is the smallest set \(Q\) such that \(P^* \subseteq Q\) and clauses 1-4 below hold:

1. \(\sigma^+ \in P^* \land [\sigma]_i^{P^*A} \in W \rightarrow \sigma^+[i/q(A)] \in Q\) (positive \(\Rightarrow\) positive)

2. \(\sigma^+ \in P^* \land [\sigma]_i^{P^*A} \notin W \rightarrow \sigma^-+[i/q(A)] \in Q\) (positive \(\Rightarrow\) negative)

3. \(\sigma^- \in P^* \land [\sigma]_i^{P^*A} = A \land \emptyset \in W \rightarrow \sigma^+[i/q(A)] \in Q\) (negative \(\Rightarrow\) positive)

4. \(\sigma^- \in P^* \land [\sigma]_i^{P^*A} = A \land \emptyset \notin W \rightarrow \sigma^-+[i/q(A)] \in Q\) (negative \(\Rightarrow\) negative)

To illustrate, consider the evaluation of \(\text{Every boy (Q}_1)\) tickled exactly three girls (Q\(_2\)) by means of \(s-exp\(_i\)\) in a model with \(B = \{b_1, b_2\}, G = \{g_1, g_2, g_3, g_4\}\) and \(T = \{(b_1, g_1), (b_1, g_2), (b_1, g_3), (b_2, g_1), (b_2, g_2), (b_2, g_3), (g_1, g_2), (g_1, g_3), (g_1, g_4), (g_2, g_1), (g_2, g_2), (g_2, g_3), (g_3, g_1), (g_3, g_2), (g_3, g_3), (g_4, g_1), (g_4, g_2), (g_4, g_3), (g_4, g_4)\}\). By the first clause of \(s-exp\(_2\)\) we can add \(\langle b_1, [Q_2]+, +\rangle, \langle b_2, [Q_2], +\rangle\) and \(\langle g_1, [Q_2], +\rangle\), and by clause 1 of \(s-exp\(_1\)\) we add \(\langle [Q_1], [Q_2], +\rangle\), since \(\{b_1, b_2, g_1\} \cap B\) is a witness set of \(\text{every boy}\). The subject-wide-scope reading of \(Q_1\) in \(V\) \(Q_2\) is true in \(M\) iff \(\langle [Q_1], [Q_2], +\rangle \in c-exp\(_1\)(\langle [Q_1], [Q_2], [V^*]\rangle)\). Clearly, if \(\langle [Q_1], [Q_2], +\rangle \in s-exp\(_2\)(\langle [Q_1], [Q_2], [V^*]\rangle)\), then \(\langle [Q_1], [Q_2], +\rangle \in c-exp\(_2\)(\langle [Q_1], [Q_2], [V^*]\rangle)\).

Importantly, the truth evaluation of this sentence in a model requires neither inferences based

\(^2\)A quantifier \(Q\) satisfies \(\text{CONS}\) iff for all domains \(E\) and for all \(A, B \subseteq E\), it holds that \(Q_E(A, B)\) iff \(Q_E(A, A \cap B)\), and \(\text{EXT}\) iff for all domains \(E\) and all \(A, B \subseteq E,\) it holds that \(Q_E(A, B)\) iff \(Q_A(A, B)\).
on negative information nor polarity reversal. On the other hand, whether Less than two boys (Q1) tickled exactly three girls (Q2) can be shown to be true by s-exp depends on the situation. If B = \{b1, b2\}, G = \{g1, g2, g3, g4\} and T = \{(b1, g1), (b1, g2), (b1, g3)\}, then \{b1, [Q2], +\} is added by s-exp2, and by s-exp1 we add \([Q1], [Q2], +\), since \{b1\} is a witness set of less than two boys. But in the situation where B = \{b1, b2\}, G = \{g1, g2, g3, g4\} and T = \{(g1, g2)\} c-exp is required: By clause 4 of c-exp2, we add the negative information that \{b1, [Q2], −\} and \{b2, [Q2], −\}, and by clause 3 of c-exp1, we add the positive information \([Q1], [Q2], +\) based on the negative information added before (polarity reversal). An important consequence of this theory is the following proposition:

**Proposition 1:** Let \(q_1(A_1) = \langle A_1, W_1 \rangle\) and \(q_2(A_2) = \langle A_2, W_2 \rangle\) be w-quantifiers with \(\emptyset \notin W_1\) and \(\emptyset \notin W_2\), and let \(P^*\) be the polarity relation of some predicate \(P \subseteq E^n\). Then \(\langle q_1(A_1), q_2(A_2), + \rangle \in \text{s-exp1}(q_1(A_1), \text{s-exp2}(q_2(A_2), P^*))\) if and only if \(\langle q_1(A_1), q_2(A_2), + \rangle \in \text{c-exp1}(q_1(A_1), \text{c-exp2}(q_2(A_2), P^*))\).

That is, if a quantified statement contains no empty-set quantifiers, then \(\langle q_1(A_1), q_2(A_2), + \rangle\) can be added by s-exp if and only if it can be added by c-exp - in other words, the evaluation of such a statement’s truth requires only inferences of positive information from positive information. If, on the other hand, a quantified statement does contain an empty-set quantifier, then it cannot be evaluated for truth by s-exp in every model, as shown above.

Based on our theory we derive the following predictions about the processing of quantified sentences. **Prediction 1:** Given that negation and polarity reversal increase processing difficulty \([2]\), we predict first that quantified statements which have to be evaluated by c-exp are more difficult to process than statements which can be evaluated by s-exp. Non-empty-set quantifiers (e.g. UE quantifiers like at least/more than n or every\(^4\) ) can be evaluated by s-exp in all models/situations. On the other hand, the evaluation of empty-set quantifiers (all DE quantifiers, among others) depends on the model: in some models s-exp suffices, in others c-exp is necessary. We predict increased processing difficulty whenever s-exp does not suffice. In particular, processing difficulty should be increased if a rule has to be applied that involves polarity reversal (clauses 2 and 3) or is based on negative information (clauses 3 and 4). The most difficult cases should be applications of clause 3 because this clause involves polarity reversal and inference from negative information. **Prediction 2:** If the comprehension of a quantified sentence involves the specification of the simplest algorithm for checking this sentence in any model, we predict not only differences in ease of verification but also differences in ease of comprehension: sentences involving no empty-set \(Q\) should be read faster than sentences involving empty set \(Q\)’s, because the former requires the specification of an algorithm involving s-exp, whereas the latter involves the complex expansion c-exp.

### 3 Experiments

**Experiment 1:** 72 German participants read simply quantified intransitive sentences of the type *Q dots are blue* manipulating \(Q\): (a) mindestens ein... (at least one, non-empty-set), (b) höchstens ein... (at most one, empty-set) and (c) weniger als zwei (less than two, empty-set). After reading the sentence, a picture was presented that showed either zero objects (0-model), one object (1-model), or three objects (3-model) of the relevant color among other objects of a different color.

\(^3\)Note that this proposition also holds for empty predicates.

\(^4\)Assuming existential import *every* can also be considered a non-empty-set quantifier since it presupposes a non-empty restriction \([3]\). Note that without this presupposition the empty set is a witness set of *every*(\(A\)) in case \(A\) is empty.
All objects (total number between 4 to 7 varying across items) in the display were elements of the restrictor set. 27 items were constructed in a $3 \times 3$ within design and distributed to nine lists using a latin square. The descriptive statistics are presented in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>0-model</th>
<th>1-model</th>
<th>3-model</th>
<th>judgment times</th>
</tr>
</thead>
<tbody>
<tr>
<td>at least one</td>
<td>0%</td>
<td>0%</td>
<td>4.9%</td>
<td>1511ms</td>
</tr>
<tr>
<td>fewer than two</td>
<td>10.3%</td>
<td>2.0%</td>
<td>0%</td>
<td>2076ms</td>
</tr>
<tr>
<td>at most one</td>
<td>22.5%</td>
<td>1.0%</td>
<td>1.0%</td>
<td>1939ms</td>
</tr>
</tbody>
</table>

Table 1: Mean judgments and judgment times in Exp. 1.

The analysis of error rates revealed that, except for the two empty-set quantifiers with 0-models, all of the conditions had error rates well below 5%. The 0-models led to 10.3% errors for less than and to even 22.6% errors for at most$^5$. In a logit mixed effects model analysis, the observed differences led to a significant quantifier $\times$ model interaction ($z = 2.42$). The increased error rates for empty-set $Q$’s when the predicate is empty are fully expected since this case requires the application of the most difficult clause of c-exp, namely clause 3, which involves polarity reversal and inference from negative information. If the predicate is not empty, then there is an $a \in E$ with $\langle a, + \rangle \in P^*$, so clause 1 or 2 can be applied. The verification times$^6$ also showed a significant interaction between quantifier and model (ANOVAs: $p_1 < .01$; $p_2 < .05$) which lends further support to our theory (prediction 1). Thus, the evaluation of 0-models proves to be a source of quantificational complexity in empty-set quantifiers even in simply quantified intransitive sentences.

Experiment 2: 72 participants read German sentences of the form $Q_1$ boys tickled $Q_2$ girls which have independently been shown to exhibit surface scope only [cf. 1]. $Q_1$ was either one of the Aristotelian quantifiers jeder (each, non-empty-set) and kein (no, empty-set) or one of the superlative quantifiers mindestens ein (at least one, non-empty-set) and höchstens

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$^5$Being well aware of the fact that these “errors” may be (at least partly) caused by pragmatic factors.

$^6$All judgment times and reading times in Exp. 1/2 were corrected for outliers by excluding RTs from the analysis that were below 100ms or above a participant’s mean RT plus 2.5sd’s.
Figure 2: Results of Experiment 2. \( \text{es} = \text{empty-set} \ Q_1; \ \text{non-es} = \text{non-empty-set} \ Q_2).\

\( \text{ein} \) (at most one, empty-set). \( Q_2 \) was either \( \text{mehr als zwei/drei} \) (more than two/three, non-empty-set) or \( \text{weniger als zwei/drei} \) (less than two/three, empty-set). \( \text{[8]} \) is a sample sentence, asterisks indicate segmentation for self-paced reading. The underlined noun phrase of the second quantifier marks the critical region since this is the earliest point at which a verification algorithm can be fully specified.

\begin{align*}
\text{(8) Mindestens ein } & \* \text{ Junge } * \text{ kitzelte } * \text{ mehr } * \text{ als } \text{ drei } * \text{ Mädchen.} \\
& \text{At least one } * \text{ boy } * \text{ tickled } * \text{ more } * \text{ than three } * \text{ girls.}
\end{align*}

Participants read these sentences in a self-paced reading experiment with moving window presentation. After each sentence, a set diagram appeared on the screen of the types shown in Figure 1 and they had to provide a truth value judgment. As in Exp. 1 there were three types of diagrams: 0-, 1- and 3-set diagrams which showed a) no, b) one or c) three boys tickling \( Q_2 \) girls, respectively. Accordingly, the experiment employed a factorial 2 (empty-set \( Q_1 \) vs. non-empty-set \( Q_1 \)) \( \times 2 \) (type of \( Q_1 \): Aristotelian vs. superlative) \( \times 2 \) (empty-set \( Q_2 \) vs. non-empty-set \( Q_2 \)) \( \times 3 \) (diagram) within design. Diagrams were constructed in such a way that across sentence conditions always the same set of pictures was used for the 0- vs 1- vs 3-set diagram. Since 0- and 3-models had to be swapped for empty-set vs. non-empty-set-\( Q_2 \)s, judgment times had to be collapsed over diagram types to allow for comparison. 72 experimental items plus 78 fillers were constructed and distributed to 24 lists in a latin square. Across the experiment, 50% of the sentences were true. Sentence-picture-pairs were presented in randomized order.
The results of Exp. 2 are for the most part consistent with our quantification theory. The observed pattern of reading times supports the distinction between s-exp and c-exp and the verification data lend support to the predicted difficulty of empty-set quantifiers when having to evaluate a situation in which the scope of the subject quantifier phrase consists of the empty set. Only the judgment times are not fully compatible with our theory.
4 Conclusions

We have presented an algorithmic theory of quantifier interpretation which predicts that the operation employed in processing empty-set quantifiers is more difficult than that for processing non-empty-set quantifiers. This prediction was mainly confirmed in two experiments that investigated the online comprehension and verification of simply and doubly quantified sentences.

In order to further disentangle complexity effects that are due to the presence of empty-set quantifiers from complexity effects due to the monotonicity of the quantifiers, we plan to extend this line of research and investigate whether non-monotonic $Q$'s which are not empty-set $Q$'s (e.g. exactly one boy or exactly three boys) are easier to process than non-monotonic empty-set $Q$'s (e.g. no boy or exactly three boys), a prediction which - to our knowledge - is unique to the present account.

References


