A Fregean Semantics for Number Words

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Abstract

This paper proposes a Fregean semantics for cardinal numbers , analysing them as properties (Chierchia 1985). A cardinal numeral can occur at the predicative type denoting a set, and at an argument type denoting the individual correlate of a set. Lexical powers like *hundred* and *thousand* denote multiplicands and are of a different type from other cardinals.

1 Cardinals as singular terms

A semantics for number words must account for numerical expressions in a number of different constructions. These include prenominal modifier position, predicate position, argument position and as 'determiners' of multiplicative cardinal heads as in the complex numerical *two hundred*. These constructions are illustrated in (1)-(4) respectively:

(1) a. *Two* cats were in the garden.
b.The *two* girls cooked a wonderful meal.
c. The guests were *two* girls.

(2) a. My reasons are *two*.
b. The children were *two*.

(3) a. *Two* plus *two* is *four*.
b. *Two* is the only even prime number

(4) *Two* hundred people stood in line.

We want to give as simple an account of cardinals as is possible, ideally deriving all these uses from a single meaning. This requires answering the following questions: (i) What is the type of the cardinal in its standard prenominal position as in (1a)? (ii) What is the relation between the cardinal in prenominal position and its use as a singular term in (3)? (iii) What is the relation between *two* and *hundred* in (4). And we would like to extend our to account for the interpretation of numericals in constructions like (5) and (6) as well.

(5) Two kilos of strawberries went into this jam.

(6) Hundreds of people stood in line.

We start by addressing the second question, and arguing in favour of treating the numerical expressions in (3) as singular terms. This is not uncontroversial. Hofweber (2005) and Ionin and

Matushansky (2006) both argue that numericals do not occur as singular terms. Hofweber argues that *two* in *two boys* is a determiner at type <<**e**,**t**>,**t**> and Ionin and Matushansky argue that it is a predicate modifier of type <<**e**,**t**>,<**e**,**t**>>. Both argue that (7a) (= (3a)) is derived from (7b) by N ellipsis. Hofweber suggests that abstract generalisations such as (7c) are derived from (7b) by 'cognitive coercion'. It is unclear exactly how cognitive coercion works, but it seems to be an extra-grammatical process of generalising from multiple specific instances to a general statement.

(7) a. Two and two make four.
b. Two things and two things make four things.
c. Two plus two makes/#make four.

However, there are a number of pieces of grammatical evidence that apparent bare singulars like *two* are ambiguous between *two (things)*, where the numerical functions as a cardinal modifier and the modified N is possibly null, and *two* as genuine singular term. First, the verb *count* is ambiguous between two meanings: 'count how many N there are' and 'name in sequence a string of natural numbers. These two uses are shown in (8a) and (8b) respectively.

(8) a. I counted thirteen (*things, people, books*).
b. I counted (up) to thirteen (**things*).
c. I counted (*up to) the books.

Count (*how many*) in (8a) takes a direct object either *two N* or *two* modifying an elliptical N. *Count* in the second sense must be followed by the P *to* and a numeral which must be bare. Further, *count* in the first sense need not have a cardinal complement, while *count (up) to* must take a numerical complement. The contrast between (8a) and (8b) correlates with the fact that 'counting how many' necessarily requires counting objects under a certain description, as discussed in Rothstein (2010). This shows up explicitly in French, where *compter* used in the sense of 'count how many' when followed by a bare numeral obligatorily appears with the clitic *en*, 'of them'. (9a) means "I counted thirteen *of them*". This clitic is impossible when *compter* is used in its second sense, as shown in (9b).

(9) a. J'*(**en**) ai compté treize.
I **of-them** AUX-PAST counted thirteen.
" I counted thirteen (of them)."
b. J'(*en) ai compté jusqu'à treize.
I AUX-PAST counted until thirteen.
"I counted up to thirteen"

Second, statements about numbers using bare cardinals cannot be rephrased using the paradigm in (7). Numbers have second order properties which do not hold of sets of objects or pluralities.

(10) a. *Two* is even / is a prime number.
b. #*Two things* are even / are a prime number.

Third, two numbers *n* and *m* do not the same as the relation as two entities with cardinalities of *n* and *m*. Numbers stand in the *bigger than/smaller than* relation, while objects with cardinality properties stand in the *more than/fewer than* relation. (11a) is equivalent to (11b), not to (11c).

(11) a. Two *is smaller than* three.
b. #Two things *are smaller than* three things.
c. Two things *are fewer than* three things.

Fourth, singular terms require singular agreement, while elliptical nouns modified by cardinals may take plural agreement.

(12) a. Four *is/*are* bigger than three.
b. Four (things) *are* more than three (things).
c. "I can't take you all in the car: Five are/is too many"

Together, these data indicate that cardinal numericals must have interpretations as singular terms, as well as an interpretation prenominally and in predicate position.

2 Cardinals as predicates

So what account should we give of cardinals in non-argument position, as in the examples (1) and (2)? The plausible options are three. They could be cardinal predicates of the same type as adjectives i.e. type <**e**,**t**> as argued in Landman (2003); they could be determiners in classical Montague style, at type <<**e**,**t**>, t> as argued in Barwise and Cooper (1981) and more recently Hofweber (2005); they could also be cardinal modifiers of type <<**e**,**t**>,<**e**,**t**>> as argued in Ionin and Matushansky (2006). A crucial factor deciding is that, as we have seen, numericals must have an interpretation at the argument type. On the assumption that there is one basic meaning for numericals, we are looking for an account in which the shift between argument type and the type licensing other uses is simple. The most straightforward account is that prenominal cardinals are predicates, since as we will see in the next section, there is an obvious way to shift between predicate and argument type. Landman (2003) gives a convincing account of cardinals as adjectives which denoting sets of plural individuals with a specific cardinality: $\{x: | x | = n\}$. This explains the fact that they can be bare predicates as in (2). Like any adjective, they shift to the modifier type in prenominal attributive position. In (1c), the sentential predicate is the NP [two_{AP} girls_N] translating as the predicate $\lambda x.\text{GIRLS}(x) \wedge | x | = 2$. Landman argues that in *the two girls,* in (1b), *two* is also a prenominal adjective. *Two girls* denotes the same property as in (1c) and is the argument of the determiner *the*. When there is no determiner, as in (1a), the cardinal adjective raises to determiner position triggering existential quantification. The DP *two girls* in (1a) denotes $\lambda P.\exists x[\text{GIRLS}(x) \wedge P(x) \wedge | x | = 2]$. Landman shows that this account correctly predicts that the predicate *two girls* will have an 'exactly 2' interpretation in (1b,1c) and an 'at least 2' reading as a DP in (1a), when it denotes a generalized quantifier.

(13) a. #The guests were two girls, and maybe even three
b. #The two girls, and maybe even three arrived late.
c. Two girls, and maybe even three, got 100 on the exam.

Accounts which suggest that numericals are born at the type determiners have a much harder time explaining how DP lowers to predicate position in (1c) and (2), especially since, as is well known generalized quantifiers such as *every girl* do not lower in this way. It is also unclear how to account for bare numerical predicates like (2) if *two* is a determiner with an elliptical NP. Hofweber (2005) discusses (14), suggesting that *seven* is a focussed determiner, but he does not give any account of how this would work.

(14) The number of planets is seven.

An extensively developed account of cardinals as predicate modifiers is presented in Ionin and Matushansky (2006). They start from two desiderata. The first is that *four* should have a single interpretation in (15a) and (15b).

(15) a. *four* cats b. *four* hundred cats

The second is that all cardinal numerical expressions are of the same type. In particular, *four* and *hundred* in (15b) should both be of the same type. They propose that all cardinal numericals are predicate modifiers at type <<**e**,**t**>,<**e**,**t**>>. In (15a) *four* modifies *cats.* In (15b) *hundred* modifies *cats* and *four* modifies *hundred cats*. (15b) has the recursive structure in (16):

(16) [four [hundred [cats]]]

A numerical *n* denotes a partition on a plural individual which has *n* parts:

(17) $\lambda P \lambda x_{pl}.\exists S\ [\ \mathrm{PARTITION}(S, x) \wedge |\ S\ | = n \wedge \forall s \in S{:}\ P(s)]$ (Ionin & Matushansky 2006: (5))

Thus, (15a) denotes a partition on the set of cats into four cells each containing a single cat (see (18b). (15b) denotes a partition on a set of pluralities of cats into 4, with each individual cell itself portioned into 100 cells each containing an individual cat. Thus the **multiplicative** effect is obtained (18c).

(18) a. *four*: $\lambda P \lambda x_{pl}.\exists S\ [\ \mathrm{PARTITION}(S, x) \wedge |\ S\ | = 4 \wedge \forall s \in S{:}\ P(s)]$
b. *four cats*: $\lambda x_{pl}.\exists S\ [\ \mathrm{PARTITION}(S, x) \wedge |\ S\ | = 4 \wedge \forall s \in S{:}\ \mathrm{CAT}(s)]$
c. *four hundred cats:* $\lambda x_{pl}.\exists S\ [\ \mathrm{PARTITION}(S, x) \wedge |\ S\ | = 4 \wedge$
$\forall s \in S{:}\ \exists S'\ \mathrm{PARTITION}\ (S', s) \wedge |\ S'\ | = 100\ \wedge \forall s' \in S'{:}\ \mathrm{CAT}(s')]$

Four hundred and four cats is analysed as an NP conjunction as in (19), with both occurrences of *four* analysed as predicate modifiers.

(19) [four [hundred [~~cats~~]]]$_{NP}$ and [four [cats]] $_{NP}$

This account leaves a number of facts unexplained. First Ionin and Matushansky predict that all possible combinations of complex cardinals should appear multiplicatively. In fact these complex cardinals are very constrained. Only a few lexically specified numerals can occur as mulitplicands in structures like (15b), as (20) shows. These include the lexical powers, *hundred* (10^2), *thousand* (10^3), *myriad* (10^4) | (10^6), and words like *dozen, score*, as shown in (20a). Other combinations such as **three four cats* or **three twenty cats* are impossible, while *twenty three cats* has only an additive reading. The minimal contrast between *twenty* and *score* rules out an extralinguistic explanation.

(20) a. three dozen/score/hundred/thousand/million cats
b. three score cats/ *three twenty cats

Second, the structure in (16) cannot explain why with lexical powers the multiplier is obligatory, while with other numbers it is impossible, as shown in (21).

(21) a.*hundred cats
b. one hundred cats
c. *one twenty cats

Third, all the lexical powers that occur in multiplicative structures like (16) can occur as approximative classifiers. No others can, except for *ten*, which arguably appears in multiplicative structures as the bound morpheme *–ty*. This indicates that multiplicands and cardinals must have different structures which both constructions exploit.

(22) a. hundreds of cats
b. thousands of cats
c. scores of cats
d. *twenties of cats.

The conclusion is that there is no good reason to treat *four* and *hundred* as being the same type, and that the semantic evidence does not support (16) or the predicate modifier theory of cardinals.

3 Numbers as properties

We aim, then to give an semantic interpretation of numericals which both account for the dual use of numbers as predicates and as bare singular arguments and which allows us to give a semantics for *four* and *hundred* which explains the data in the previous section.

We adopt (and adapt) Chierchia's (1985) property theory, which follows Frege (1892) in assuming that predicates have two modes of interpretation, an unsaturated mode in which they are predicated of arguments, and a saturated mode in which they can be arguments of predicates.

Property theory associates with predicates an applicative interpretation at type <**e**,**t**>, and a corresponding property-correlate at the type of individuals, $\boldsymbol{\pi}$. Two operations $^{\cap}$ and $^{\cup}$ switch between them. For the applicative interpretation $\lambda x.\mathrm{WISE}(x)$, the individual property correlate, WISDOM is $^{\cap}\lambda x.\mathrm{WISE}(x)$, as in (23a). (A morphological expression of nominalisation is not necessary. *Blue* has an applicative use in *Her eyes are blue* and a saturated use in *Blue is my favourite colour.)*

(23) a. $[\![\mathrm{wise}_{<\mathbf{e},\mathbf{t}>}]\!] = \lambda x.\mathrm{WISE}(x)$
b. $[\![\mathrm{wise}_{\pi}]\!] = {}^{\cap}\lambda x.\mathrm{WISE}(x)$
c. ${}^{\cup\cap}\lambda x.\mathrm{WISE}(x) = \lambda x.\mathrm{WISE}(x)$

For cardinal numerals, we start out with a standard modifier interpretation at type <**e**,**t**>, as in (24a) , with the cardinality function defined as in (24b) and *x* ranging over plural individuals.

(24) a. $[\![\mathrm{four}_{<\mathbf{e},\mathbf{t}>}]\!] = \lambda x.|x| = 4$
b. $|x| = n \leftrightarrow |\{y: y \sqsubseteq_{\mathrm{ATOMIC}} x\}| = n$

(25) a. $[\![\text{four guests}]\!] = \lambda x.\mathrm{GUESTS}(x) \wedge |x| = 4$
b. $[\![\text{the guests are four}]\!] = \lambda x.|x| = 4\ (\sigma\{x: \mathrm{GUESTS}(x)\})$
$= |\sigma\{x: \mathrm{GUESTS}(x)\}| = 4$

The singular term *four* is type n. It denotes the individual property correlate of the set in (24a)

(26) $^{\cap}(\lambda x.|x| = 4)$.

The central equation which defines numbers is (27):

(27) $n = {}^{\cap}\lambda x.|x| = n$ (for numbers in the domain of type n)

Second order properties at type <**n**,**t**> such as *is a prime number* apply only to numerals at type **n**.

Lexical powers, unlike the simple numerals illustrated in (24) are of type <**n**,<**e**,**t**>> and combine with a numerical expression at type **n**, denoting a number to yield a cardinal predicate:

(28) a. $[\![\text{hundred}_{<\mathbf{n},<\mathbf{e},\mathbf{t}>>}]\!] = \lambda n\lambda x. |x| = 100 \times n$
b. $[\![\text{two hundred}_{<\mathbf{e},\mathbf{t}>>}]\!] = \lambda x. |x| = 100 \times 2$
c. $[\![\text{two hundred}_{\mathbf{n}}]\!] = 200 = {}^{\cap}(\lambda x. |x| = 200)$.

(29) $[\![\text{two hundred cats}]\!] = \lambda x.\text{CATS}(x) \wedge |x| = 100 \times 2$

Lexical powers are similar to measure expressions such as *kilo* which combine with numerals on their singular term interpretation at type **n** to give a measure predicate $\lambda n\lambda x.\ \text{MEAS}_{weight}=<n, \text{KILO}>$, in examples such as (5). (Landman 2004, Rothstein 2009). Lexical powers and measure expressions, are thus construction in which bare cardinals are used at type **n**.

Once this analysis is given, various ways of deriving complex numerals are available. One way is as follows. We derive *two hundred and four* at type **n** from (30a) together with the expressions in (24a) and (28c) shifted to type **n** via $^{\cap}$.

(30) a. $[\![\text{and}_{<\mathbf{n},<\mathbf{n},\mathbf{n}>>}]\!] = \lambda m\lambda n.\ n+m$
b. $[\![\text{two hundred and four}_{\mathbf{n}}]\!] = \lambda m\lambda n.\ m+n\ (200)$
$= \lambda n.\ 200+n\ (4)$
$= 204$
c. $[\![\text{two hundred and four}_{<\mathbf{e},\mathbf{t}>}]\!] = {}^{\cup}204 = \lambda x. |x| = 204$

'+' is defined in terms of the sum relation. It maps two numbers *n* and *m* onto the cardinality of the sum of two non-overlapping entities y and z, where y has cardinality *m* and z has cardinality *n*. This is given in (31). $\oplus$ and $\sqcup$ are the standard overlap and sum functions respectively. *Two plus three* denotes the number associated with the cardinality of the entity x, where x is the result of summing y, an entity with two atomic parts, with z, an entity with three atomic parts, if y and z do not overlap.

(31) $+_{<\mathbf{n},<\mathbf{n},\mathbf{n}>>} = \lambda m\lambda n.{}^{\cap}\lambda x.\ \forall y \in {}^{\cup}m:\ \forall z \in {}^{\cup}n:\ \neg(y \oplus z) \rightarrow |\sqcup\{y,z\}| = |x|$

'$\times$' can be given a similar interpretation, using the operation DISJOINT, which maps a set X onto X' $\subseteq$ X, such that no two members of X' overlap. $\times$ combines with *n* and *m* and yields the number of atomic parts of the sum of *n* non-overlapping entities each with cardinality *m.*

(32) $\times_{<\mathbf{n},<\mathbf{n},\mathbf{n}>>} = \lambda m\lambda n.{}^{\cap}\lambda x.\ \forall z \in \text{PLURAL}(\text{DISJOINT}({}^{\cup}m) \wedge |z| = n:$
$|\{x':\forall y \sqsubseteq z: x' \sqsubseteq_{\text{ATOM}} y\}| = |x|$

Two times three denotes the number of atomic parts of two non-overlapping entities each with three atomic parts, i.e. the number 6.

This analysis is Fregean in two senses. First, cardinals denote properties at both the applicative and argument types. As applicative, unsaturated expressions they denote cardinality properties. As saturated, argument expressions they denote numbers. This is a direct instantiation of Frege's insight (Frege 1892) that a property has 'two modes of presentation', one unsaturated in which it applied to an argument to form a sentence, and one saturated, in which it can itself be the subject of a predication. Second Frege argued that as an object, a number *n* denotes the equivalence class of sets with cardinality *n* (Frege 1884).This is captured directly. Frege treated *four* as denoting the equivalence class in (33a), i.e. the set of sets with cardinality 4, sets with four members. We have defined the meaning of the cardinality predicate as a property of plural individuals (24a). Since a plural individual with cardinality *n* is the sum of a set with *n* atomic parts (24b), the one can be reduced to the other, for example by the **PRED** function, defined in (33b), which maps sets of sets with cardinality *n* onto sets of plural individuals with cardinality *n*, the meaning of the cardinal predicate.

(33) a. $\{X: |X| = 4\}$
b. For $Y = \{X: |X| = n\}$: $\mathbf{PRED}(Y) = \{x: \exists X \in Y: x = \sqcup X\}$
c. $\mathbf{PRED}(\{X: |X| = 4\})$ $=$ $\{x: \exists X \in Y: x = \sqcup X\} =$
$\{x: |x| = 4\}$

4 Lexical powers as classifiers

We conclude by looking briefly at lexical powers in their classifer use. All lexical powers, i.e. numbers at type <**n**,<**e**,**t**>> occur as approximatives. As already noted, they must be marked plural (34a), they may not be preceded by a numerical (34b), and, they have the syntax of classifiers: they must be followed by *of* and a bare plural (34c):

(34) a. hundreds of cats/hundred of cats
b. *two hundreds of cats
c. hundreds of cats/*hundreds cats

Only lexical powers occur in this construction. (As noted above, *ten* counts as a lexical power, although as a multiplicand it only occurs in its bound form *–ty*, as in *twen-ty*, *thir-ty* etc.)

Like measure classifiers, approximative classifiers can occur with definite determiners preferably if the nominal is modified by a relative clause:

(35) a. the hundreds of cats (#that I saw in the garden)
b. the ten kilos of flour (#that should have been delivered this morning)
c. The hundreds of rabbits (that you promised me I should see) never appeared.

Since only lexical powers can be used as classifiers in this way (see the examples in (22)), we assume that the operation APPROX deriving approximative readings must exploit the <**n**,<**d,t**>> type of lexical powers. The morphological operation associated with the operation is pluralization: *hundred* become *hundreds* and so on. We assume the semantic operation is APPROX as in (36), which derives *hundreds* from *hundred*:

(36) $\text{APPROX}([\![\text{hundred}_{<\mathbf{n},<\mathbf{e},\mathbf{t}>}]\!]) = \text{APPROX}(\lambda n \lambda x. |x| = n \times 100)$
$= \lambda x. \exists n[n \geq 2 \wedge |x| \geq n \times 1000]$

The APPROX operation does three things: it existentially quantifies over the *n* argument, it changes the ‘=’ to ‘≥’ and it adds the clause ‘$n \geq 2$’. $\text{APPROX}([\![\text{hundred}_{<\mathbf{n},<\mathbf{e},\mathbf{t}>}]\!])$ gives the set of individuals which have a cardinality which is greater than $100 \times n$, where *n* is greater than or equal to 2, i.e. which have a cardinality greater than 200. Since classifiers are of type <<**e**,**t**>,<**e**,**t**>>, (36) shifts to the predicate modifier type to apply to the denotation of *cats.*

(37) $\lambda P \lambda x. \exists n[n \geq 2 \wedge |x| \geq n \times 100 \wedge P(x)]$ $(\lambda x.\text{CATS}(x))$
$= \lambda x. \exists n[n \geq 2 \wedge |x| \geq n \times 100 \wedge \text{CATS}(x)]$

Approximators can stack as in *hundreds of thousands of cats*. Function composition composes *hundred* and *thousand*, as in (38a), and APPROX applies to the whole expression, marking every lexical power plural.

(38) a. *hundred* $\circ$ *thousand* $= \lambda n\lambda x.|x| = n \times 100 \circ \lambda n\lambda x.|x| = n \times 1000$
$= \lambda n\lambda x.|x| = ({}^{\cap}(\lambda y.|y| = n) \times 100) \times 1000$

b. $\text{APPROX}(\lambda n\lambda x.|x| = ({}^{\cap}(\lambda y.|y| = n) \times 100) \times 1000) =$
$\lambda x.\exists n\,[n \geq 2 \wedge |x| = ({}^{\cap}(\lambda y.|y| = n) \times 100) \times 1000]$

c. *hundreds of thousands of cats*:
$\lambda x.\exists n\,[n \geq 2 \wedge |x| = ({}^{\cap}(\lambda y.|y| = n) \times 100) \times 1000 \wedge \text{CATS}(x)]$

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