Lattices and topology

Guram Bezhanishvili
Mamuka Jibladze

ESSLLI 2008
20th European Summer School in Logic, Language and Information
4–15 August 2008
Freie und Hansestadt Hamburg, Germany

Programme Committee. Enrico Franconi (Bolzano, Italy), Petra Hendriks (Groningen, The Netherlands), Michael Kaminski (Haifa, Israel), Benedikt Löwe (Amsterdam, The Netherlands & Hamburg, Germany) Massimo Poesio (Colchester, United Kingdom), Philippe Schlenker (Los Angeles CA, United States of America), Khalil Sima'an (Amsterdam, The Netherlands), Rineke Verbrugge (**Chair**, Groningen, The Netherlands).

Organizing Committee. Stefan Bold (Bonn, Germany), Hannah König (Hamburg, Germany), Benedikt Löwe (**chair**, Amsterdam, The Netherlands & Hamburg, Germany), Sanchit Saraf (Kanpur, India), Sara Uckelman (Amsterdam, The Netherlands), Hans van Ditmarsch (**chair**, Otago, New Zealand & Toulouse, France), Peter van Ormondt (Amsterdam, The Netherlands).

`http://www.illc.uva.nl/ESSLLI2008/`
`esslli2008@science.uva.nl`

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

ESSLLI 2008 is organized by the Universität Hamburg under the auspices of the *Association for Logic, Language and Information* (FoLLI). The *Institute for Logic, Language and Computation* (ILLC) of the *Universiteit van Amsterdam* is providing important infrastructural support. Within the Universität Hamburg, ESSLLI 2008 is sponsored by the Departments *Informatik*, *Mathematik*, *Philosophie*, and *Sprache, Literatur, Medien I*, the *Fakultät für Mathematik, Informatik und Naturwissenschaften*, the *Zentrum für Sprachwissenschaft*, and the *Regionales Rechenzentrum*. ESSLLI 2008 is an event of the *Jahr der Mathematik 2008*. Further sponsors include the *Deutsche Forschungsgemeinschaft* (DFG), the Marie Curie Research Training Site GLoRiClass, the European Chapter of the Association for Computational Linguistics, the *Hamburgische Wissenschaftliche Stiftung*, the Kurt Gödel Society, Sun Microsystems, the Association for Symbolic Logic (ASL), and the European Association for Theoretical Computer Science (EATCS). The official airline of ESSLLI 2008 is Lufthansa; the book prize of the student session is sponsored by *Springer Verlag*.

Guram Bezhanishvili and Mamuka Jibladze

Lattices and topology

Course Material. 20th European Summer School in Logic, Language and Information (ESSLLI 2008), Freie und Hansestadt Hamburg, Germany, 4–15 August 2008

Lattices and Topologies

An introductory course for ESSLLI'08

by

Guram Bezhanishvili and Mamuka Jibladze

The aim of this course is to provide the basics of two relatively new branches of mathematics **Lattice Theory** and **Topology**, which play an important role in developing the algebraic and topological semantics for non-classical logics. They have their origins in the works of two famous German mathematicians and close friends **Richard Dedekind** (1831 – 1916) and **Georg Cantor** (1845 – 1918).

Lattices are special partially ordered sets possessing suprema and infima of all finite subsets. They encode the algebraic behavior of the entailment relation and such basic logical connectives as conjunction and disjunction, which results in an adequate *algebraic semantics* for a variety of logical systems.

On the other hand, a *topology* on a set X is a collection of subsets of X—called *open sets*—containing $\emptyset$, X and closed under finite intersections and arbitrary unions. A typical example of a topological space is the real line $\mathbf{R}$ with the topology generated by the open intervals of $\mathbf{R}$. In topological spaces it makes sense to talk about points being in a neighborhood of a given point. This provides means to talk about properties that hold locally at a given point, which opens up a door for a semantics of non-classical logics called the *topological semantics*. In several instances, such as the **intuitionistic logic** or large classes of **modal logics**, the topological semantics is an extension of the *relational semantics*, which is more familiar for the ESSLLI audience.

Lattice theory and topology are closely connected. For one, the collection of open subsets of a topological space always forms a lattice. Moreover, well-behaved lattices, such as *distributive lattices*, can be represented as sublattices of the lattice of open subsets of a topological space. The logical significance of these representation theorems lies in the fact that they are essentially equivalent to results about completeness of intuitionistic and modal logics with respect to the topological semantics.

This course is dedicated to several topological representation theorems for distributive lattices and related structures. We also discuss how these representation theorems yield topological completeness results for large classes of logical systems. The course will consist of five 90 minute lectures. Below we give a short outline of each of the five lectures.

Lecture 1: Lattices: partial orders, lattices, complete lattices, lattices as algebras, filters and ideals, distributive laws, Birkhoffs characterization of distributive lattices.

Lecture 2: Duality for finite distributive lattices: join-prime elements, prime filters, correspondence between join-prime elements and prime filters, duality between finite distributive lattices and finite posets. The infinite case: first attempts to represent an infinite distributive lattice, lack of join-prime elements, prime filters, prime ideals, Stone's lemma, a representation of infinite distributive lattices.

Lecture 3: Topologies: topological spaces, open and closed sets, interior and closure operators, separation axioms, Hausdorff spaces, compact spaces, spectral spaces, Stone spaces. Priestley spaces: ordered topological spaces, Priestley separation axiom, Priestley spaces. Priestley duality: Priestley representation theorem, distributive lattice homomorphisms and continuous order-preserving maps, Priestley duality.

Lecture 4: Bitopological duality: bitopological spaces, an axiomatization of the class of bitopological spaces obtained from Priestley spaces, Priestley spaces from bitopological spaces, bitopological duality. Spectral duality: spectral spaces, how they are obtained from pairwise Stone spaces, de Groot dual, pairwise Stone spaces from spectral spaces, spectral duality.

Lecture 5: Topological completeness theorems: intuitionistic propositional calculus **IPC**, Heyting algebras as algebraic models of **IPC**, topological representation of Heyting algebras as a consequence of the spectral duality, topological completeness of **IPC**, topological completeness of some superintuitionistic logics, **S4** and some other modal systems, Gödel's translation of **IPC** into **S4**, topological completeness of **S4** and some of its extensions.

Relevant literature

There is an ample amount of literature available on lattice theory and topology. Below we provide a short selection of sources relevant to this course.

Lattices and order:

For an elementary exposition consult the first few chapters of:

Introduction to Lattices and Order - B. A. Davey and Hilary A. Priestley (second edition, 2002 Cambridge University Press)

A Course in Universal Algebra - Stanley N. Burris and H.P. Sankappanavar:
`http://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html`

Notes on lattice theory - J. B. Nation:
http://www.math.hawaii.edu/~jb/lat1-6.pdf (Chapters 1-6),
http://www.math.hawaii.edu/~jb/lat7-12.pdf (Chapters 7-12).

The following two books are standard references:

Lattice Theory - Garrett Birkhoff (third edition, 1967 AMS)

General Lattice Theory - George A. Grätzer (second edition, 1998 Birkhäuser)

Topology:

The following three are the standard textbooks on general topology:

General Topology - John L. Kelley (1975 Birkhäuser)

General Topology - Stephen Willard (2004 Courier Dover Publications)

Topology - James R Munkres (second edition, 2000 Prentice Hall)

For online sources consult:

Topology Course Lecture Notes by Aisling McCluskey and Brian McMaster, at the Topology Atlas: http://at.yorku.ca/i/a/a/b/23.htm

Elementary Topology: Math 167 - Lecture Notes by Stefan Waner:
http://people.hofstra.edu/Stefan_Waner/RealWorld/pdfs/Topology.pdf

General Topology - Jesper M. Møller:
http://www.math.ku.dk/~moller/e03/3gt/notes/gtnotes.pdf

What is topology? - Neil Strickland:
http://neil-strickland.staff.shef.ac.uk/Wurble.html

Some basic topological terminology and notation from NoiseFactory Science Archives:
http://noisefactory.co.uk/maths/topology.html#Topologicalterminologyandnotation

Duality theory:

An elementary introduction to the Priestley duality can be found in:

Introduction to Lattices and Order - B. A. Davey and Hilary A. Priestley (second edition, 2002 Cambridge University Press)

A more sophisticated and in-depth coverage of duality theory (including the spectral duality) can be found in:

Stone spaces - P. T. Johnstone (1982 Cambridge University Press)

Continuous Lattices and Domains - Gerhard Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott (2003 Cambridge University Press)

Bitopologies:

A nice introduction to bitopological spaces is:

Bitopological Spaces, Compactifications and Completions - Sergio Salbany (1974 University of Cape Town)

The following article is a recent survey on bitopological spaces:

Asymmetry and duality in topology - Ralph Kopperman (Topology and its Applications, Volume 66, Issue 1, 8 September 1995, Pages 1-39)

Completeness theorems for intuitionistic and modal logics:

For topological completeness theorems for intuitionistic and modal logics we refer to:

Intuitionistic logic and modality via topology - Leo Esakia (Annals of Pure and Applied Logic, Volume 127, Issues 1-3, June 2004, Pages 155-170)

Handbook of Spatial Logics, Marco Aiello, Ian Pratt-Hartmann, Johan van Benthem eds. (2007 Springer)