From Beth to van Benthem: Possibilities for Intuitionistic, Classical, and Modal Logic

Workshop on the Future of Logic

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On Semantics for Intuitionistic Logic

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E. W. Beth. "Semantic Construction of Intuitionistic Logic," *Mededelingen der Koninklijke Nederlandse Akademie van Wetenschappen*, 1956.

D. van Dalen. "How to Glue Analysis Models," *JSL*, 1984.

A. Troesltra and D. van Dalen. *Constructivism in Mathematics*, 1988.

On Possibility Semantics for Classical Logic

J. van Benthem. "Possible World Semantics for Classical Logic," Technical Report ZW-8018, Dept. of Mathematics, Rijksuniversiteit, Groningen, 1981.

J. van Benthem. "Partiality and Nonmonotonicity in Classical Logic," *Logique et Analyse*, 1986.

J. van Benthem. *A Manual of Intensional Logic*, Part III, 1988.

On Possibility Semantics for Modal Logic

I. L. Humberstone. "From Worlds to Possibilities," *JPL*, 1981.

W. H. Holliday. "Partiality and Adjointness in Modal Logic," *AiML*, 2014, and several manuscripts in progress (email wesholliday@berkeley.edu for drafts).

Two Paths to Possibility Semantics for Modal Logic

1. From Beth/Kripke Semantics for Intuitionistic Logic
   1.1 Go Classical
   1.2 Go Modal
2. From Relational Semantics for Modal Logic

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After sketching these two paths, I'll say something about

- The Program of Possibility Semantics.

Information Models

We begin with *information models* $\mathcal{M} = \langle S, \geqslant, \pi \rangle$ (cf. Frank Veltman's dissertation), i.e., Kripke models for intuitionistic PL:

- S is a nonempty set;
- $\geqslant$ is a partial order on S;
- $\pi\colon \mathrm{At} \times S \nrightarrow \{0,1\}$;

persistence: if $\pi(p, \mathbf{X})\downarrow$ and $\mathbf{X}' \geqslant \mathbf{X}$, then $\pi(p, \mathbf{X}') = \pi(p, \mathbf{X})$.

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Although these authors think of the partial order $\geqslant$ as a branching *temporal* relation, arguably this specific idea is not essential.

What is essential is the more abstract idea that for partial information states X and X', the relationship $X' \geqslant X$ indicates that X' is a possible extension (refinement, enrichment, etc.) of X, so X' settles every issue that X does and perhaps more.

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Not only **intuitionistic logic**, but also **classical logic** (Benthem 1981, Benthem 1986, Garson 2013) and various **modal logics** (Humberstone 1981, Veltman 1985, Holliday 2014, Yalcin 2014) can be given natural semantics using such structures.

Kripke Forcing

Given an information model $\mathcal{M} = \langle S, \geqslant, \pi \rangle$ with $\mathbf{X} \in S$ and $\varphi \in \mathcal{L}_{\mathsf{PL}}$, we define $\mathcal{M}, \mathbf{X} \Vdash_k \varphi$ as follows:

- $\mathcal{M}, \mathbf{X} \Vdash_k p$ iff $\pi(p, \mathbf{X}) = 1$;
- $\mathcal{M}, \mathbf{X} \Vdash_k \neg\varphi$ iff $\forall \mathbf{X}' \geqslant \mathbf{X}$: $\mathcal{M}, \mathbf{X}' \nVdash_k \varphi$;
- $\mathcal{M}, \mathbf{X} \Vdash_k \varphi \wedge \psi$ iff $\mathcal{M}, \mathbf{X} \Vdash_k \varphi$ and $\mathcal{M}, \mathbf{X} \Vdash_k \psi$;
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Lemma (Persistence)

If $\mathcal{M}, \mathbf{X} \Vdash_k \varphi$ and $\mathbf{X}' \geqslant \mathbf{X}$, then $\mathcal{M}, \mathbf{X}' \Vdash_k \varphi$.

Two Main Ideas

Go Classical:

Intuitionistic semantics should not have a monopoly on the use of partial information states. Classical semantics should also be able to use information models, albeit in a different way.

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Cf. Johan's (2009) "The Information in Intuitionistic Logic."

Gödel Translation

Recall the Gödel translation:

- $p^G = \neg\neg p$;
- $(\neg\varphi)^G = \neg\varphi^G$;
- $(\varphi \wedge \psi)^G = \varphi^G \wedge \psi^G$;
- $(\varphi \vee \psi)^G = \neg(\neg\varphi^G \wedge \neg\psi^G)$;
- $(\varphi \to \psi)^G = (\varphi^G \to \psi^G)$.

Theorem (Gödel 1933)

For all $\varphi \in \mathcal{L}_{\mathsf{PL}}$, φ is a theorem of classical propositional logic iff φ^G is a theorem of intuitionistic propositional logic.

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Theorem (Gödel 1933)

For all $\varphi \in \mathcal{L}_{\mathsf{PL}}$, φ is a theorem of classical propositional logic iff φ^G is a theorem of intuitionistic propositional logic.

Idea: let's implement the Gödel translation at the semantic level.

First, recall:

- $\mathcal{M}, \mathbf{X} \Vdash_k \neg\neg p$ iff $\forall \mathbf{X}' \geqslant \mathbf{X}\ \exists \mathbf{X}'' \geqslant \mathbf{X}'$: $\pi(p, \mathbf{X}'') = 1$.

$p^G = \neg\neg p$, so we will define:

- $\mathcal{M}, \mathbf{X} \Vdash_c p$ iff $\forall \mathbf{X}' \geqslant \mathbf{X}\ \exists \mathbf{X}'' \geqslant \mathbf{X}'$: $\pi(p, \mathbf{X}'') = 1$.

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$(\neg\varphi)^G = \neg\varphi^G$, $(\varphi \wedge \psi)^G = \varphi^G \wedge \psi^G$, $(\varphi \to \psi)^G = \varphi^G \to \psi^G$, so the $\neg$, $\wedge$, and $\to$ clauses for $\Vdash_c$ will be the same as for $\Vdash_k$.

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so the $\neg$, $\wedge$, and $\rightarrow$ clauses for $\Vdash_c$ will be the same as for $\Vdash_k$.

Finally, observe:

- $\mathcal{M}, \mathbf{X} \Vdash_k \neg(\neg\varphi \wedge \neg\psi)$ iff $\forall \mathbf{X}' \geqslant \mathbf{X}\ \exists \mathbf{X}'' \geqslant \mathbf{X}'$:
  $\mathcal{M}, \mathbf{X}'' \Vdash_k \varphi$ or $\mathcal{M}, \mathbf{X}'' \Vdash_k \psi$.

$(\varphi \vee \psi)^G = \neg(\neg\varphi^G \wedge \neg\psi^G)$, so we will define:

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Classical Forcing

Given an information model $\mathcal{M} = \langle S, \geqslant, \pi \rangle$ with $\mathbf{X} \in S$ and $\varphi \in \mathcal{L}_{\mathsf{PL}}$, we define $\mathcal{M}, \mathbf{X} \Vdash_c \varphi$ as follows:

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Cofinality (Refinability)

Given an information model $\mathcal{M} = \langle S, \geqslant, \pi \rangle$ with $\mathbf{X} \in S$ and $\varphi \in \mathcal{L}_{\mathsf{PL}}$, we define $\mathcal{M}, \mathbf{X} \Vdash_c \varphi$ as follows:

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Lemma (Cofinality)

If $\forall \mathbf{X}' \geqslant \mathbf{X}\ \exists \mathbf{X}'' \geqslant \mathbf{X}'\ \mathcal{M}, \mathbf{X}'' \Vdash_c \varphi$, then $\mathcal{M}, \mathbf{X} \Vdash_c \varphi$.

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Lemma (Refinability)

If $\mathcal{M}, \mathbf{X} \nVdash_c \varphi$, then $\exists \mathbf{X}' \geqslant \mathbf{X}$: $\mathcal{M}, \mathbf{X}' \Vdash_c \neg\varphi$.

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Theorem

Classical propositional logic is sound and complete with respect to the class of information models with the $\Vdash_c$ semantics.

Kripke Forcing vs. Classical Forcing

So the differences are:

- $\mathcal{M}, \mathbf{X} \Vdash_k p$ iff $\pi(p, \mathbf{X}) = 1$;
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vs.

- $\mathcal{M}, \mathbf{X} \Vdash_c p$ iff $\forall \mathbf{X}' \geqslant \mathbf{X}\ \exists \mathbf{X}'' \geqslant \mathbf{X}'$: $\pi(p, \mathbf{X}'') = 1$;
- $\mathcal{M}, \mathbf{X} \Vdash_c (\varphi \vee \psi)$ iff $\forall \mathbf{X}' \geqslant \mathbf{X}\ \exists \mathbf{X}'' \geqslant \mathbf{X}'$:
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We get a subtly different view of the contrast between classical and intuitionistic perspectives if we consider Beth semantics...

Beth Forcing (à la van Dalen 1984)

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So the differences are:

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- $\mathcal{M}, \mathbf{X} \Vdash_b \varphi \vee \psi$ iff $\forall$ paths $\mathcal{P}$ through $\mathbf{X}$ $\exists \mathbf{X}' \in \mathcal{P}$:
$$\mathcal{M}, \mathbf{X}' \Vdash_b \varphi \text{ or } \mathcal{M}, \mathbf{X}' \Vdash_b \psi;$$

vs.

- $\mathcal{M}, \mathbf{X} \Vdash_c p$ iff $\forall \mathbf{X}' \geqslant \mathbf{X}$ $\exists \mathbf{X}'' \geqslant \mathbf{X}'$: $\pi(p, \mathbf{X}'') = 1$;
- $\mathcal{M}, \mathbf{X} \Vdash_c \varphi \vee \psi$ iff $\forall \mathbf{X}' \geqslant \mathbf{X}$ $\exists \mathbf{X}'' \geqslant \mathbf{X}'$:
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The intuitionist wants it to be inevitable that one's information will include p (or decide between φ and ψ), whereas the classicist is satisfied if it is always possible to extend one's information to include p (or decide between φ and ψ)—if p (or the decision between φ and ψ) is *always available*, rather than *inevitable*.

Classical Forcing

Given an information model $\mathcal{M} = \langle S, \geqslant, \pi \rangle$ with $\mathbf{X} \in S$ and $\varphi \in \mathcal{L}_{\mathsf{PL}}$, we define $\mathcal{M}, \mathbf{X} \Vdash_c \varphi$ as follows:

- $\mathcal{M}, \mathbf{X} \Vdash_c p$ iff $\forall \mathbf{X}' \geqslant \mathbf{X}\ \exists \mathbf{X}'' \geqslant \mathbf{X}'$: $\pi(p, \mathbf{X}'') = 1$;
- $\mathcal{M}, \mathbf{X} \Vdash_c \neg\varphi$ iff $\forall \mathbf{X}' \geqslant \mathbf{X}$: $\mathcal{M}, \mathbf{X}' \nVdash_c \varphi$;
- $\mathcal{M}, \mathbf{X} \Vdash_c \varphi \wedge \psi$ iff $\mathcal{M}, \mathbf{X} \Vdash_c \varphi$ and $\mathcal{M}, \mathbf{X} \Vdash_c \psi$;
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Final step toward propositional possibility semantics: let's push the $\forall\exists$ pattern even deeper, from $\Vdash$ to a condition on models.

Propositional Possibility Models

Recall our information models $\mathcal{M} = \langle S, \geqslant, \pi \rangle$:

- S is a nonempty set;
- $\geqslant$ is a partial order on S;
- $\pi \colon \mathrm{At} \times \wp(S) \nrightarrow \{0, 1\}$;

persistence: if $\pi(p, \mathbf{X})\downarrow$ and $\mathbf{X}' \geqslant \mathbf{X}$, then $\pi(p, \mathbf{X}') = \pi(p, \mathbf{X})$.

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Johan's (1981) "Possible World Semantics for Classical Logic" studies possibility models for classical first-order logic.

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refinability: if $\pi(p, \mathbf{X})\uparrow$, then $\exists \mathbf{Y}, \mathbf{Z} \geqslant \mathbf{X}$: $\pi(p, \mathbf{Y}) = 0$, $\pi(p, \mathbf{Z}) = 1$.

Humberstone (1981) uses the stronger *refinability* condition.

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Given a propositional possibility model $\mathcal{M} = \langle S, \geqslant, \pi \rangle$ with $\mathbf{X} \in S$ and $\varphi \in \mathcal{L}_{\mathsf{PL}}$, we define $\mathcal{M}, \mathbf{X} \Vdash \varphi$ as follows:

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- $\mathcal{M}, \mathbf{X} \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, \mathbf{X} \Vdash \varphi$ and $\mathcal{M}, \mathbf{X} \Vdash \psi$;
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- $\mathcal{M}, \mathbf{X} \Vdash \varphi \rightarrow \psi$ iff $\forall \mathbf{X}' \geqslant \mathbf{X}$: if $\mathcal{M}, \mathbf{X}' \Vdash \varphi$ then $\mathcal{M}, \mathbf{X}' \Vdash \psi$.

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Lemma (Persistence)

If $\mathcal{M}, \mathbf{X} \Vdash \varphi$ and $\mathbf{X}' \geqslant \mathbf{X}$, then $\mathcal{M}, \mathbf{X}' \Vdash \varphi$.

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Lemma (Cofinality)

If $\forall \mathbf{X}' \geqslant \mathbf{X}\ \exists \mathbf{X}'' \geqslant \mathbf{X}'\ \mathcal{M}, \mathbf{X}'' \Vdash \varphi$, then $\mathcal{M}, \mathbf{X} \Vdash \varphi$.

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Theorem

Classical propositional logic is sound and complete w.r.t. the class of propositional possibility models with the above semantics.

Going Modal

Recall our information models $\mathcal{M} = \langle S, \geqslant, \pi \rangle$:

- S is a nonempty set;
- $\geqslant$ is a partial order on S;
- π: At $\times \wp(S) \nrightarrow \{0, 1\}$;

persistence: if $\pi(p, \mathbf{X})\downarrow$ and $\mathbf{X}' \geqslant \mathbf{X}$, then $\pi(p, \mathbf{X}') = \pi(p, \mathbf{X})$.

A *propositional possibility model* is a model as above that satisfies:

cofinality: if $\forall \mathbf{X}' \geqslant \mathbf{X}\ \exists \mathbf{X}'' \geqslant \mathbf{X}'\ \pi(p, \mathbf{X}'') = 1$, then $\pi(p, \mathbf{X}) = 1$.

Q: what kind of structure should we add to these for modal logic?

The philosophical man in the street occasionally speaks of so-and-so's 'belief-world': the world as so-and-so believes it to be. And philosophers rightly protest at such terminology since ... a person may believe a disjunction without believing either disjunct, so that there really is no such thing as *the* world as so-and-so believes it to be. Rather, in such cases as the ones just envisaged, what we have is that the disjunction holds at every one of so-and-so's belief compatible worlds, the one disjunct holding at some, the other at others ....

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The philosophers' objection to the 'belief-world' locution is well taken, and it may be respected while at the same time justice is done to the man in the street's idea that (not quite putting it in his own terms) there should be a single entity such that truth with respect to it coincides with being believed by so-and-so; the entity won't be a possible world, but, instead, a possibility. (Humberstone 1981, 334)

Functional Possibility Models

In my AiML 2014 paper, I introduced the following modification of Humberstone's (1981) original modal possibility models.

A *functional possibility model* is a tuple $\mathcal{M} = \langle S, \geqslant, \{f_a\}_{a\in I}, \pi\rangle$:

- S is a set with a distinguished element $\bot_{\mathcal{M}}$; $\boldsymbol{S} = S - \{\bot_{\mathcal{M}}\}$;
- $\geqslant$ is a partial order on $\boldsymbol{S}$; $\mathbf{X} \between \mathbf{Y}$ iff $\exists \mathbf{Z} \in \boldsymbol{S}$: $\mathbf{Z} \geqslant \mathbf{X}$ and $\mathbf{Z} \geqslant \mathbf{Y}$;

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persistence: if $\pi(p, \mathbf{X})\downarrow$ and $\mathbf{X}' \geqslant \mathbf{X}$, then $\pi(p, \mathbf{X}') = \pi(p, \mathbf{X})$;

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modal analogue of *persistence*...

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cofinality: if $\forall \mathbf{X}' \geqslant \mathbf{X}\ \exists \mathbf{X}'' \geqslant \mathbf{X}'\ \pi(p, \mathbf{X}'') = 1$, then $\pi(p, \mathbf{X}) = 1$;

f-persistence: if $\mathbf{X}' \geqslant \mathbf{X}$, then $f_a(\mathbf{X}') \geqslant f_a(\mathbf{X})$;

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f-persistence: if $\mathbf{X}' \geqslant \mathbf{X}$, then $f_a(\mathbf{X}') \geqslant f_a(\mathbf{X})$;

f-refinability: if $\mathbf{Y} \geqslant f_a(\mathbf{X})$, then $\exists \mathbf{X}' \geqslant \mathbf{X}\ \forall \mathbf{X}'' \geqslant \mathbf{X}'$: $\mathbf{Y} \between f_a(\mathbf{X}'')$.

Functional Possibility Semantics

Given a functional possibility model $\mathcal{M} = \langle S, \geqslant, \{f_a\}_{a \in I}, \pi \rangle$ with $\mathbf{X} \in S$ and $\varphi \in \mathcal{L}_{\mathsf{ML}}$, we define $\mathcal{M}, \mathbf{X} \Vdash \varphi$ as follows:

- $\mathcal{M}, \bot_{\mathcal{M}} \Vdash \varphi$ for all $\varphi \in \mathcal{L}_{\mathsf{ML}}$;
- $\mathcal{M}, \mathbf{X} \Vdash p$ iff $\pi(p, \mathbf{X}) = 1$;
- $\mathcal{M}, \mathbf{X} \Vdash \neg\varphi$ iff $\forall \mathbf{X}' \geqslant \mathbf{X}$: $\mathcal{M}, \mathbf{X}' \nVdash \varphi$;
- $\mathcal{M}, \mathbf{X} \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, \mathbf{X} \Vdash \varphi$ and $\mathcal{M}, \mathbf{X} \Vdash \psi$;
- $\mathcal{M}, \mathbf{X} \Vdash \Box_a \varphi$ iff $\mathcal{M}, f_a(\mathbf{X}) \Vdash \varphi$.

Humberstone (1981): "there should be a single entity such that truth with respect to it coincides with being believed by so-and-so."

As usual, we can define $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi := \neg(\varphi \wedge \neg\psi)$, $\Diamond_a \varphi := \neg\Box_a\neg\varphi$.

Definition (Possibilization)

Given a relational world model $\mathfrak{M} = \langle W, \{R_a\}_{a \in I}, V \rangle$, define the *powerset possibilization* of $\mathfrak{M}$, $\mathfrak{M}^\star = \langle S, \geqslant, \{f_a\}_{a \in I}, \pi \rangle$:

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1. $S = \wp(W)$ and $\bot_{\mathfrak{M}^\star} = \emptyset$;
2. $X \geqslant Y$ iff $X \subseteq Y$;
3. $f_a(X) = R_a[X] = \{y \in W \mid \exists x \in X \colon xR_a y\}$;
4. $\pi(p, X) = \begin{cases} 1 & \text{if } \forall x \in X \colon V(p, x) = 1; \\ 0 & \text{if } \forall x \in X \colon V(p, x) = 0 \text{ and } X \neq \emptyset; \\ \text{undef.} & \text{otherwise.} \end{cases}$

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Lemma (From Worlds to Possibilities)

For any relational world model $\mathfrak{M} = \langle W, \{R_a\}_{a\in I}, V\rangle$:

1. $\mathfrak{M}^\star$ is a *functional possibility model*;
2. $\mathfrak{M}^\star, X \Vdash \varphi$ iff for all $x \in X$, $\mathfrak{M}, x \vDash \varphi$.

Axiomatization

Axioms: as usual, D is $\Box_a\varphi \to \neg\Box_a\neg\varphi$, T is $\Box_a\varphi \to \varphi$, 4 is $\Box_a\varphi \to \Box_a\Box_a\varphi$, B is $\neg\varphi \to \Box_a\neg\Box_a\varphi$, and 5 is $\neg\Box_a\varphi \to \Box_a\neg\Box_a\varphi$.

Theorem (Soundness and Strong Completeness)

For any subset of $\{\mathrm{D}, \mathrm{T}, 4, \mathrm{B}, 5\}$, the extension of the modal logic **K** (in its polymodal version) with that set of axioms is sound and strongly complete with respect to the class of functional possibility models satisfying the associated condition for each axiom:

D axiom: for all $\mathbf{X}$, $f_a(\mathbf{X}) \neq \bot$;

T axiom: for all $\mathbf{X}$, $\mathbf{X} \geqslant f_a(\mathbf{X})$;

4 axiom: for all $\mathbf{X}$, $f_a(f_a(\mathbf{X})) \geqslant f_a(\mathbf{X})$;

B axiom: for all $\mathbf{X}, \mathbf{Y}$, if $\mathbf{Y} \geqslant f_a(\mathbf{X})$ then $\exists \mathbf{X}' \geqslant \mathbf{X}$: $\mathbf{X}' \geqslant f_a(\mathbf{Y})$;

5 axiom: for all $\mathbf{X}, \mathbf{Y}$, if $\mathbf{Y} \geqslant f_a(\mathbf{X})$, then $\exists \mathbf{X}' \geqslant \mathbf{X}$: $f_a(\mathbf{X}') \geqslant f_a(\mathbf{Y})$.

A Simpler Completeness Proof

The infinitary baggage that comes with possible worlds—maximally consistent sets and Lindenbaum's Lemma—are not involved in the completeness proof, which takes the domain of the canonical model to simply be the set of sets of formulas (modulo equivalence).

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(Incidentally, as a result, the proof easily generalizes to uncountable languages without the use of the Axiom of Choice.)

A Simpler Completeness Proof

The infinitary baggage that comes with possible worlds—maximally consistent sets and Lindenbaum's Lemma—are not involved in the completeness proof, which takes the domain of the canonical model to simply be the set of sets of formulas (modulo equivalence).

(Incidentally, as a result, the proof easily generalizes to uncountable languages without the use of the Axiom of Choice.)

In fact, to prove the *weak* completeness of many standard extensions of **K**, we can take the domain of the canonical model to simply be the set of individual formulas (modulo equivalence).

The Problem of the Irrelevant Extension

Van Benthem (1988, p. 78): "There is something inelegant to an ordinary Henkin argument. One has a consistent set of sentences S, perhaps quite small, that one would like to see satisfied semantically. Now, some arbitrary *maximal* extensions S^+ of S is to be taken to obtain a model (for S^+, and hence for S)—but the added part $S^+ - S$ plays no role subsequently. We started out with something partial, but the method forces us to be total."

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This "problem of the 'irrelevant extension'" (van Benthem 1981, p. 1) is solved by possibility semantics for classical and modal logic.

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Banning Worlds

Humberstone and other philosophers discuss the idea that possibilities (unlike total worlds) can always be further refined:

- $\mathcal{M}$ is *infinitely ascending* iff $\forall \mathbf{X} \in \boldsymbol{S}\ \exists \mathbf{X}' \in \boldsymbol{S}$: $\mathbf{X}' > \mathbf{X}$.

I also consider the condition that each possibility settles only finitely many atomic facts (we are assuming At is infinite):

- $\mathcal{M}$ is *locally finite* iff $\forall \mathbf{X} \in \boldsymbol{S}$: $\{p \in \mathsf{At} \mid \pi(p, \mathbf{X})\downarrow\}$ is finite.

For completeness w.r.t to locally finite models, the proof depends more on the particular logic, so we treat epistemic/doxastic logics:

Theorem (Completeness Cont.)

Each $\mathbf{L} \in \{\mathbf{K}, \mathbf{KD}, \mathbf{T}, \mathbf{K4}, \mathbf{KD4}, \mathbf{K45}, \mathbf{KD45}, \mathbf{S4}, \mathbf{S5}\}$ is complete w.r.t the class of *locally finite* and *infinitely ascending* possibility models with the conditions associated with **L**'s axioms:

D axiom: for all $\mathbf{X}$, $f_a(\mathbf{X}) \neq \bot$;

T axiom: for all $\mathbf{X}$, $\mathbf{X} \geqslant f_a(\mathbf{X})$;

4 axiom: for all $\mathbf{X}$, $f_a(f_a(\mathbf{X})) \geqslant f_a(\mathbf{X})$;

B axiom: for all $\mathbf{X}, \mathbf{Y}$, if $\mathbf{Y} \geqslant f_a(\mathbf{X})$ then $\exists \mathbf{X}' \geqslant \mathbf{X}$: $\mathbf{X}' \geqslant f_a(\mathbf{Y})$;

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This is proved by constructing a canonical possibility model whose points are (equivalence classes of) *individual finite formulas*.

Two Parts of the Program

1. show how familiar modal languages can be given possibility semantics instead of world semantics, revealing

   1.1 which of the assumptions about the nature and structure of possibilities that are built into world models are (un)necessary to obtain familiar logics for these languages and

   1.2 which alternative assumptions about possibilities, assumptions that are inconsistent with those built into world models, are compatible with familiar logics for these languages;

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    1.2 which alternative assumptions about possibilities, assumptions that are inconsistent with those built into world models, are compatible with familiar logics for these languages;

2. identify extended languages for which possibility semantics allows us to obtain new logics that we cannot obtain with standard world semantics, or with possibilities as sets of worlds, revealing the limitations of world-based modeling.

Extended Languages

For an example of the point about expanded languages, suppose that we add to our language the Kripke and Beth disjunctions:

- $\mathcal{M}, \mathbf{X} \Vdash \varphi \vee_k \psi$ iff $\mathcal{M}, \mathbf{X} \Vdash \varphi$ or $\mathcal{M}, \mathbf{X} \Vdash \psi$;
- $\mathcal{M}, \mathbf{X} \Vdash \varphi \vee_b \psi$ iff for $\forall$ paths $\mathcal{P}$ through $\mathbf{X}$ $\exists \mathbf{X}' \in \mathcal{P}$: $\mathcal{M}, \mathbf{X}' \Vdash \varphi$ or $\mathcal{M}, \mathbf{X}' \Vdash \psi$.

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Then as Ivano Ciardelli observed, the Kreisel-Putnam principle

$$\big(\alpha \to_k (\varphi \vee_k \psi)\big) \to_k \big((\alpha \to_k \varphi) \vee_k (\alpha \to_k \psi)\big),$$

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And the principle $\varphi \vee_b \neg\varphi$ is valid over possibilizations of world models. But it is not valid over all possibility models.

Further Directions

- Possibility models for first-order ML (Harrison-Trainor 2014)
- Possibility models for epistemic modals (Yalcin 2014)
- Possibility models for dynamic epistemic logic (Holliday 2014)
- Possibility models for (un)awareness in epistemic logic
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- Connections between possibility and situation semantics
- and more ...

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Stay tuned or join the project!

Thank You

&

Thanks to Johan
for inspiring so many of us!

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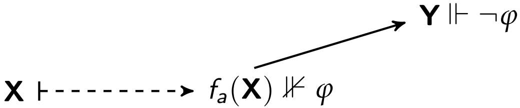

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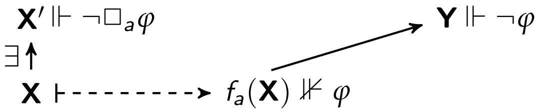

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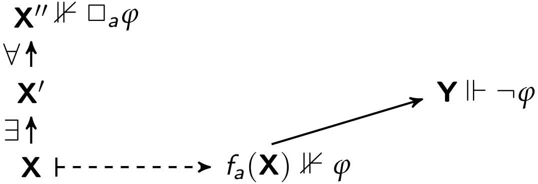

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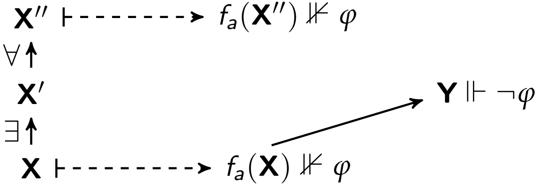

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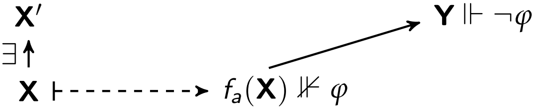

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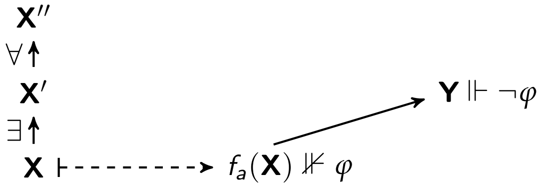

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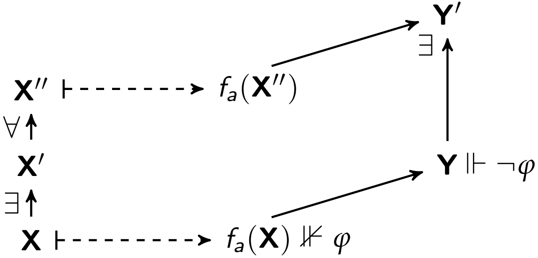

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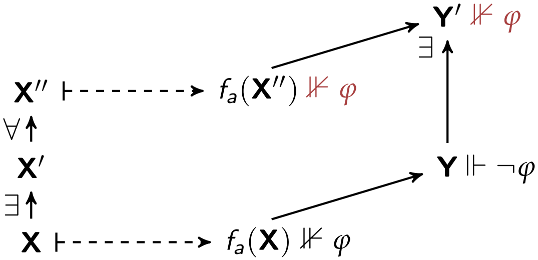

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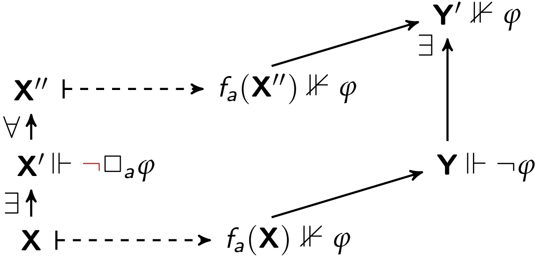

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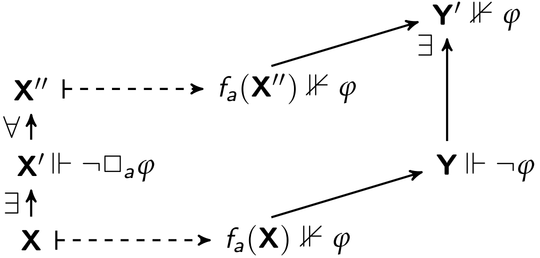

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f-refinability: if $\mathbf{Y} \geqslant f_a(\mathbf{X})$, then $\exists \mathbf{X}' \geqslant \mathbf{X}\ \forall \mathbf{X}'' \geqslant \mathbf{X}'$: $\mathbf{Y} \Vvee f_a(\mathbf{X}'')$.

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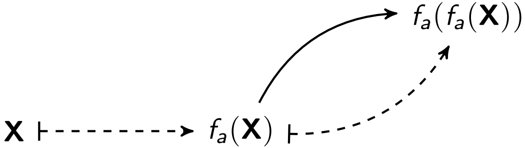

B axiom - $\neg\varphi \to \Box_a \neg\Box_a \varphi$

-

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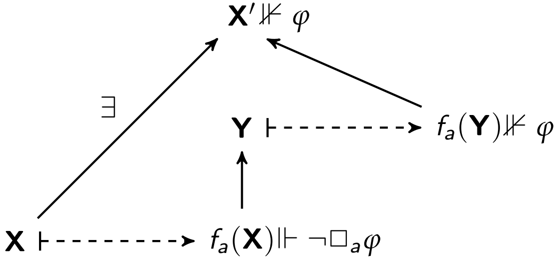

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