Johan van Benthem and Löb's Logic

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Descent

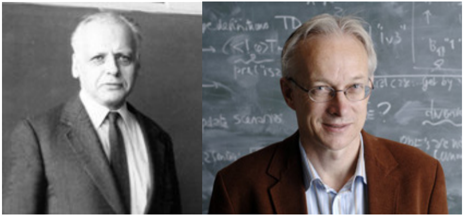

Figure: Martin Löb & Johan van Benthem

Johan van Benthem was Löb's phd student in the period 1973–1977. His phd thesis was: *Modal Correspondence Theory*.

Johan and Provability Logic

Throughout his career Johan took an interest in provability logic, not only in his publications but also in letters and conversations.

In his published work, there are two major things. His early result on the closed fragment of provability logic and his later work on the relation between Löb's Logic and the μ-calculus. There is ongoing work flowing from that last contribution —more on that at another occasion.

We will discuss the early contribution and what followed it, since this offers a unique opportunity to make some historical remarks.

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Löb's Logic

Löb's Logic GL is the modal logic given by the following principles.

L1. $\vdash (\Box\phi \wedge \Box(\phi \to \psi)) \to \Box\psi$,

L2. $\vdash \Box\phi \to \Box\Box\phi$,

L3. $\vdash \Box(\Box\phi \to \phi) \to \Box\phi$,

L4. $\vdash \phi \quad \Rightarrow \quad \vdash \Box\phi$.

Löb's Logic is the logic of provability in Σ_1-sound theories extending Elementary Arithmetic (Solovay 1976). We have soundness already for S^1_2. (We demand that the axiom set be given by, say, a Δ^b_1-formula.)

The closed fragment of Löb's Logic is simply the special case GL^0 of GL on zero variables.

Sample formula: $\Box(\Box\Box\bot \to \Diamond\top) \vee \Box\Diamond\Box\top$.

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Friedman's 35th Problem

In 1975, Harvey Friedman's publishes a list of 102 problems in mathematical logic in JSL. Problem 35 is as follows.
Consider the set of all GL^0-formulas that are arithmetically valid if we interpret $\Box$ as provability in a given theory. Is this set decidable?

For Σ_1-sound extensions of S^1_2 this set is precisely the theorems of GL_0. So the question becomes: is GL_0 decidable?

Friedman's question was *in part* a prelude to the question (†) concerning the completeness of Löb's Logic for arithmetical interpretations. This question was answered by Robert Solovay in 1976. Note however that the characterization of the closed fragment works for a wider range of theories.

Why didn't Harvey ask (†)?

Friedman's 35th Problem: Solution

- *Johan van Benthem* (1974): Solution in unpublished note. The work was directed at solving (†). It was triggered by a colloquium organized by Dick de Jongh and Craig Smoryński. Johan thought that the solution was insignificant compared to Solovay's result and never published. The manuscript of Johan's work is somewhere in his attic.
- *Roberto Magari* (1975): What is the free Magari algebra on 0 generators. Magari was probably *not* trying to solve (†).
- *George Boolos* (1976): George was trying to solve (†). He had the result before he saw Friedman's question.
- *Claudio Bernardi & Franco Montagna* (1976): When they saw Friedman's question, they realized that the solution was known to 'their group'. Their proof was not published because George submitted it to JSL just before them. The manuscript of their work is possibly lost.

Normal Form Theorem

Let α, β range over $\omega^+ := \{0, 1, \ldots \infty\}$. We define:

- $\Box^0 \bot := \bot$
- $\Box^{n+1} \bot := \Box \Box^n \bot$
- $\Box^\infty \bot = \top$

Every closed formula of GL^0 is equivalent to a formula of the form:

$$\bigwedge_i (\Box^{\alpha_i} \bot \to \Box^{\beta_i} \bot), \text{ where } \beta_i < \alpha_i.$$

Reduction of our sample formula:

$$\begin{aligned}
\vdash \Box(\Box\Box\bot \to \Diamond\top) \vee \Box\Diamond\Box\top \quad &\leftrightarrow \quad \Box\neg(\Box\Box\bot \wedge \Box\bot) \vee \Box\neg\Box\neg\Box\top \\
&\leftrightarrow \quad \Box\neg\Box\bot \vee \Box\neg\Box\bot \\
&\leftrightarrow \quad \Box\bot
\end{aligned}$$

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Ignoring artificially tailored cases, we do not know the full provability logic of any extension of HA except for the case of PA. However, we do know the closed fragments of the provability logics of:

- HA,
- $\mathsf{HA}^* := \mathsf{HA} + \{A \to \Box_{\mathsf{HA}^*} A \mid A \in \mathcal{L}\}$,
- $\mathsf{HA} + \mathsf{MP}$ and $\mathsf{HA} + \mathsf{MP}_{\mathsf{PR}}$,

These matters were resolved in papers by AV in 1985, 1994, 2002 and 2008.

The description of the constructive cases reveals the true nature of the solution in the classical case.

Great open question: what happens if we add versions of Church's Thesis?

Degrees of Falsity 1

We consider $\omega^+ := \{0, 1, \ldots \infty\}$ as *degrees of falsity*. 0 is the falsest falsity (or $\bot$) and ∞ is the truest falsity or $\top$. We let α, β range over degrees of falsity. The language $\mathcal{D}$ is:

- $\phi ::= \alpha \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi)$.

The theory Basic is given by intuitionistic propositional logic with 0 in the role of $\bot$ and ∞ in the role of $\top$, plus the principles $\vdash \alpha \rightarrow \beta$, for $\alpha \leq \beta$.

We consider the following extensions of Basic.

- Stronglöb := Basic + $\{((\alpha \rightarrow \beta) \rightarrow \beta) \mid \beta < \alpha\}$,
- Stable := Basic + $\{\neg\neg\,\alpha \rightarrow \alpha \mid \alpha \in \omega^+\}$,
- Classical := Basic + $\{\alpha \vee \neg\,\alpha \mid \alpha \in \omega^+\}$.

Stronglöb is the theory of the unique Heyting Algebra on ω^+ with the usual ordering.

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Degrees of Falsity 2

Basic corresponds to HA, Stronglöb corresponds to HA^*, Stable corresponds to HA $+$ MP and Classical corresponds to PA.

We need only one more concept: let Λ be a theory in $\mathcal{D}$. We define $\alpha_\Lambda(\phi)$ as the largest α such that $\Lambda \vdash \alpha \to \phi$. Under reasonable conditions α_Λ is always defined.

Let Λ be given. We translate the language of modal logic without propositional variables into $\mathcal{D}$:

- $\mathsf{tr}_\Lambda(\bot) := 0$, $\mathsf{tr}_\Lambda(\top) := \infty$
- tr_Λ commutes with the propositional connectives,
- $\mathsf{tr}_\Lambda(\Box\phi) := \alpha_\Lambda(\mathsf{tr}_\Lambda(\phi)) + 1$.

Degrees of Falsity 3

We define $\mathsf{AL}(\Lambda) := \{\phi \mid \Lambda \vdash \mathsf{tr}(\phi)\}$.

- $\mathsf{A}(\Lambda)$ extends $i\mathsf{GL}_0$ the intuitionistic version of GL_0 (under reasonable assumptions).
- $\mathsf{AL}(\mathsf{Basic})$ is the closed fragment of the provability logic of HA. Etcetera. These results can be viewed as *box-elimination*.
- Of $\mathsf{AL}(\mathsf{Stronglöb})$ and $\mathsf{AL}(\mathsf{Classic})$ we have axiomatizations: to wit the obvious ones. Of the other two we don't.
- If Λ_0 and Λ_1 are different, then there is a ϕ such that $\Lambda_0 \vdash \phi$ and $\Lambda_1 \nvdash \phi$ (under reasonable assumptions).

	Bas	Stro	Sta	Cla
$\Box(\Box\bot \vee \neg\Box\bot) \to \Box\Box\bot$	$+$	$+$	$+$	$-$
$\Box(\neg\neg\Box\bot \to \Box\bot) \to \Box\Box\bot$	$+$	$+$	$-$	$-$
$\Box\neg\neg\Box\bot \to \Box\Box\bot$	$+$	$-$	$+$	$+$

Overview

Spinn-Off

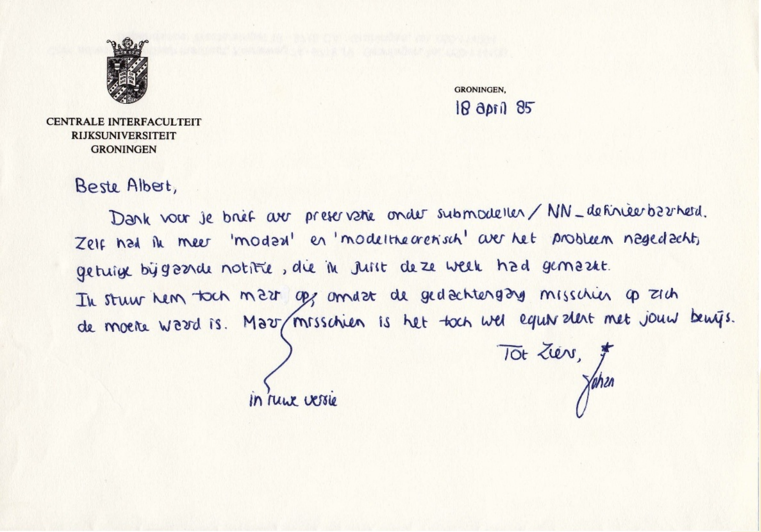
One of the subjects that turned out to be relevant in the study of intuitionistic provability logic and its closed fragments is the formula class NNIL: no nestings of implications to the left.

In 1985, I corresponded with Johan about the question whether NNIL formulas were precisely the formulas preserved under taking sub-Kripke-models. On April 16, I mailed Johan my argument that this was indeed the case. On April 18, I received a neat notition by Johan with a beautiful, more 'mathematical' proof of the same.

Johan's proof resulted in a paper by Johan, Dick de Jongh, Gerard Renardel and me about NNIL-formulas in 1995.

NNIL formulas were further studied by Fan Yang, Dick de Jongh, Nick Bezhanishvili in the context of frames. They were studied by Carlos Cotrini and Yuri Gurevic in the context of infon logic.

April 18, 1985



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GRONINGEN, 18 april 85

Beste Albert,

Dank voor je brief over preservatie onder submodellen / NN_definieerbaarheid. Zelf had ik meer 'modaal' en 'modeltheoretisch' over het probleem nagedacht, getuige bijgaande notitie, die ik juist deze week had gemaakt. Ik stuur hem toch maar op, (in ruwe versie) omdat de gedachtengang misschien op zich de moeite waard is. Maar misschien is het toch wel equivalent met jouw bewijs.

Tot ziens,
Johan

Figure: April 18, 1985

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Other Results 1

1991 Petr Hájek and Vítěslav Švejdar characterize the closed fragment of ILF: ILF normal forms are GL normal forms.

1992 Albert Visser characterizes the closed fragment of the provability and interpretability logic of $\mathrm{I}\Delta_0 + \Omega_1$ with a constant for Exp.

1993 The closed fragment of Japaridze's logic GLP is characterized by Konstantin Ignatiev.

1993 Sergei Artemov shows that the elementary theory of the 0-generated GL-algebra is decidable (equivalent to Buechi's WS1S), published in a joint paper with Lev Beklemishev.

2004 Start of Lev Beklemishev's program to characterize proof theoretic ordinals in terms of the closed fragment of Japaridze's Logic: *worms*.

Other Results 2

2005 Joost Joosten characterizes the closed fragment of the provability and interpretability logic of PRA with a constant for $\mathrm{I}\Sigma_1$. Joost refers to earlier work by Lev in 1996 on the provability logic of PRA with a constant for $\mathrm{I}\Sigma_1$.

2011 Joost Joosten and Félix Bou: the closed fragment of IL is pspace hard.

2012 Vedran Čačic and Mladen Vuković: give examples of IL-formulas without normal forms.

2013 Vedran Čačic and Vjekoslav Kovać: more than 93% of the closed IL-formulas have GL-equivalents.

2013 The closed fragment of GL is polytime decidable (Rybakov-Chagrov). For GLP with finitely many modalities it is still in P, however for GLP itself it is PSpace-complete (Pakhomov).

2014 Pakhomov shows that the elementary theory of the 0-generated subalgebra of GLP is decidable. However, the semilattice of worms has an undecidable elementary theory.