Royal Netherlands Academy of Arts and Sciences (KNAW)

Master Class

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Epistemic Foundations of Game Theory

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Assignments for Lecture 1

1. (Representing knowledge). There are ten cards. Each card has two numbers written on it. The cards are as follows: {2,2} (i.e. this card has the number 2 written on it twice), {2,3} (this card has the numbers 2 and 3 written on it), {2,4}, {2,5}, {3,3}, {3,4}, {3,5}, {4,4}, {4,5}, {5,5}. A referee shows all the cards to Mr. Sum and Ms. Parity, then shuffles them and picks one without showing it to them. He then announces loudly to S and P (so that it becomes commonly known between them) that he will tell S **privately** the sum of the two numbers on the card and will tell P **privately** whether the product of the two numbers is even or odd. The referee then proceeds to do that (whispers in S's ears the sum and whispers in P's ear whether the product is even or odd). Assume throughout that any statements made by the players are truthful; indeed it is common knowledge between S and P that they always tell the truth.

(a) Represents the possible states of knowledge of S and P using information partitions.

(b) At 1:00 pm P tells S: "I don't know what card the referee picked". Does this statement convey any information to S? If Yes, what is the information? If No, why not?

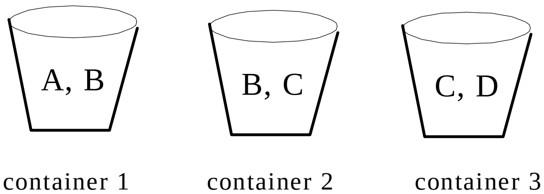
(c) At 1:10 pm S tells P: "I don't know what card the referee picked". Does this statement convey any information to P? If Yes, explain what the information is and represent P's possible states of information by means of a new information partition. If No, why not?

(d) At 1:20 pm S asks P: "do you know whether the number that was communicated to me (i.e. the sum of the two numbers) is even or odd?". P replies: "No, I don't." Hearing this, S says "Then I now know what the card is". P replies "Well, if you know what the card is then I still don't know but I can narrow down the possibilities to two". What cards does P have in mind as possibilities?

(e) At this stage you and Albert enter the room. The referee shows all the cards to you and Albert. He then tells both of you: "I picked one of these cards (I forget which one now) and I secretly told S the sum of the two numbers and told P whether the product was odd or even; then P and S had the following conversation …" (he plays back the recorded conversation between P and S). Albert hears this and proposes the following bet to you: "we'll ask Mr S what the card was; if the sum of the numbers on the card was even you give me $10, while if the sum was odd then I'll give you $x". For what value of x should you accept the bet?

2. (Knowledge and common knowledge).There are two players, 1 and 2, and a referee. The referee publicly tells the players the following:

> «There are three containers. Container 1 has two cards in it, one with the letter A and the other with the letter B; container 2 has two cards, one with the letter B and the other with the letter C, and container 3 has two cards, one with the letter C and the other with the letter D.



> First I will put you in separate rooms. Then, in a third room, I will secretly choose one container, take the two cards from that container and put them in two separate envelopes. I will shuffle the envelopes and give one to each of you (for example, if I pick container 1, then player 1 might end up being given the envelope with the letter A - so that player 2 gets B - or the envelope with the letter B - so that player 2 ends up with the letter A). Each of you will then secretly open your own envelope and look at your own card.»

(a) Represent the possible states of information of the two players (when they open their own envelopes) by means of a set of states and information partitions.

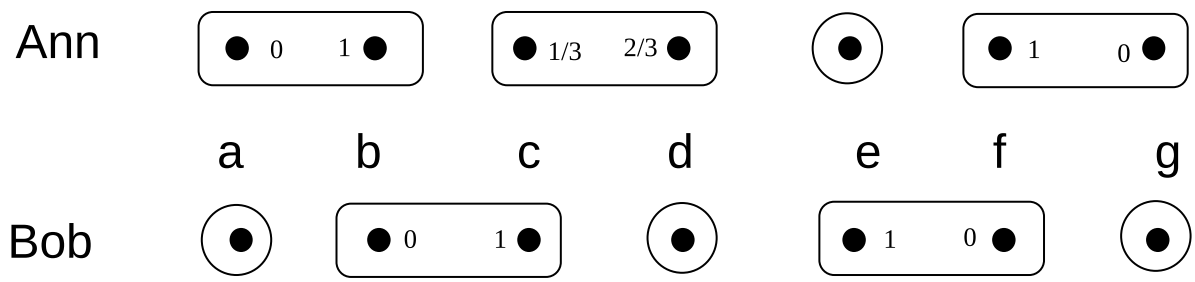
(b) Let E be the event "player 2 gets a B card". Find the events K_1E (player 1 knows E) and K_2K_1E (player 2 knows that player 1 knows E).

(c) Find the common knowledge partition.

(d) Suppose that the referee, at the very beginning (before sending them to their separate rooms), also told the players the following:
«After you look at your own card, I want you to write an ordered pair of letters (for example, the pair (A,B)) on a piece of paper. I will then collect the pieces of paper and if (1) you both wrote the same ordered pair and (2) one of the two letters (it doesn't matter which one) matches the one that was in the envelope given to player 1, then I will give you \$100 each.»
Before they are taken to their different rooms the players can talk to each other and agree on a plan. What plan will guarantee that they will each get \$100, no matter what cards they end up being given?

(e) Suppose now that the referee, at the beginning, tells the players that he will choose container 1 with probability $\frac{1}{6}$, container 2 with probability $\frac{1}{3}$ and container 3 with probability $\frac{1}{2}$. Then, after choosing the container, the referee will toss a fair coin and give the first letter to player 1 if the coin comes up Heads, otherwise he will give her the second letter (and, of course, give the other letter to player 2).
(e.1) Suppose that player 1 opens his envelope and tells you that it is not an A. What is the probability that player 2 finds a C in her envelope? [Use Bayes' rule].
(e.2) Suppose that (not very cleverly) the players agreed that – no matter what cards they got – they would both write the pair (A,B). What is the probability, before the game starts, that they will end up getting \$100 each?

3. (Belief and common belief). Consider the following interactive knowledge/belief structure (the numbers inside each cell are probabilities; for singleton cells the probability of the corresponding state is obviously 1). Let ω be a state and E an event (set of states). We say that at ω: (1) Ann *knows* E if the cell of Ann's partition that contains ω is a subset of E and (2) Ann *believes* E if she assigns probability 1 to E. Similarly for Bob.



(a) Find the common knowledge partition.

(b) If F is an event, let $K_i F$ denote the event that individual $i \in \{\text{Ann, Bob}\}$ knows F and CKF the event that F is common knowledge between Ann and Bob.

Let $E = \{a, c, d\}$. Determine the following events:

(b.1) $K_{\text{Ann}}E$. (b.2) $K_{\text{Bob}}E$. (b.3) $K_{\text{Ann}}K_{\text{Ann}}E$. (b.4) $K_{\text{Ann}}K_{\text{Bob}}E$

(b.5) $K_{\text{Bob}}K_{\text{Ann}}E$. (b.6) CKE.

(c) Find the individual belief relations and the common belief relation. [Hint: if $\mathcal{B}_{\text{Ann}}$ denotes Ann's belief relation then $\omega \mathcal{B}_{\text{Ann}} \omega'$ if and only if at ω Ann assigns positive probability to ω'. Recall that the common belief relation is the transitive closure of $\mathcal{B}_{\text{Ann}} \cup \mathcal{B}_{\text{Bob}}$.]

(d) If F is an event, let $B_i F$ denote the event that individual $i \in \{\text{Ann, Bob}\}$ believes F and CBF the event that F is commonly believed by Ann and Bob.

As before, let $E = \{a, c, d\}$. Determine the following events:

(d.1) $B_{\text{Ann}}E$. (d.2) $B_{\text{Bob}}E$. (d.3) $B_{\text{Ann}}B_{\text{Bob}}E$. (d.4) $B_{\text{Bob}}B_{\text{Ann}}E$. (d.5) CBE.

4. (Models of games). Consider the following two-player strategic-form game, where the payoffs are von Neumann-Morgenstern payoffs.

Player 2

Player 1		e	f	g
	A	6 , 0	2 , 0	0 , 3
	B	2 , 3	0 , 6	2 , 3
	C	0 , 0	8 , 3	4 , 6
	D	0 , 9	2 , 0	6 , 6

Let S_1 = {A, B, C, D}, S_2 = {e, f, g} and $S = S_1 \times S_2$. A *probabilistic model* of this game consists of

(1) a set of states Ω,

(2) a partition of Ω for each player (representing what the player knows),

(3) for each cell of each information partition a probability distribution whose support is contained in that cell (representing the beliefs of the corresponding player), and

(4) a function $\sigma : \Omega \to S$ (describing each state in terms of what the players do).

Given a model, a state ω and a player i, let $I_i(\omega)$ denote the cell of i's partition, that contains ω, $p_{i\omega}$ the probability distribution over $I_i(\omega)$, $\sigma_i(\omega)$ i's choice at ω and $\sigma_{-i}(\omega)$ the choice of the other player at ω (so that $\sigma(\omega) = (\sigma_i(\omega), \sigma_{-i}(\omega))$). A model of a game satisfies the following constraints:

(1) every player knows his own choice, that is, for every player i and for every two states $\omega, \omega' \in \Omega$, if $\omega' \in I_i(\omega)$ then $\sigma_i(\omega) = \sigma_i(\omega')$, and

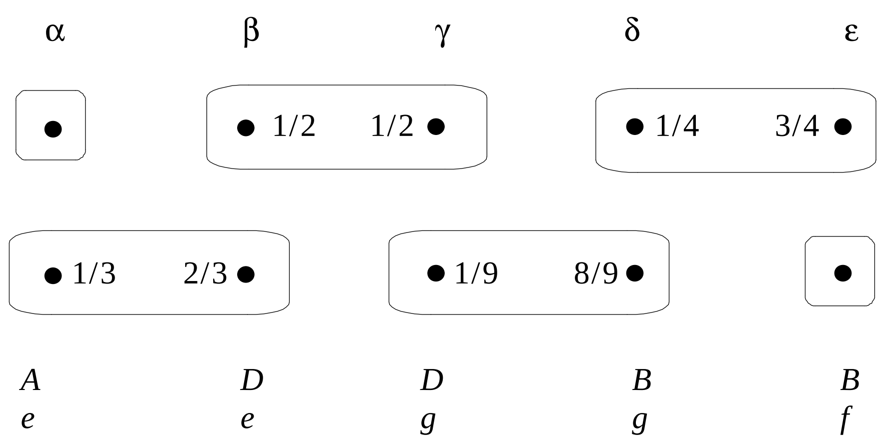
(2) every player knows his own beliefs, that is, for every player i and for every two states $\omega, \omega' \in \Omega$, if $\omega' \in I_i(\omega)$ then $p_{i\omega} = p_{i\omega'}$.

Given a state ω and a player i we say that i *is rational at* ω if i's choice at ω maximizes his expected utility at ω, that is, if

$$\sum_{\omega' \in \Omega} u_i(s, \sigma_{-i}(\omega'))\, p_{i\omega}(\omega') \geq \sum_{\omega' \in \Omega} u_i(x, \sigma_{-i}(\omega'))\, p_{i\omega}(\omega') \quad \text{for every } x \in S_i$$

where $s = \sigma_i(\omega)$ and $u_i : S \to \mathbb{R}$ is player i's von Neumann-Morgenstern payoff function.

Consider the following model of the above game:



(a) Let R_1 denote the event that player 1 is rational and similarly for R_2. Find R_1 and R_2.

(b) Find the events K_1R_2 (player 1 knows that player 2 is rational), K_2R_1, $K_1K_2R_1$, $K_2K_1R_2$, $K_1K_2K_1R_2$, $K_2K_1K_2R_1$, $CK(R_1 \cap R_2)$ (it is common knowledge that both players are rational).

5. (Models of games). Consider the following strategic-form game:

		Player 2 d	e	f
Player 1	A	4, 1	2, 2	0, 0
	B	1, 0	1, 0	1, 6
	C	0, 2	3, 1	4, 0

(a) Construct a model of this game that satisfies the following properties:

(1) there are four states,

(2) there is a state at which rationality is common knowledge.

(b) In your model, at how many states is there common knowledge of rationality?