Royal Netherlands Academy of Arts and Sciences (KNAW)

Master Class

Amsterdam, February 8th, 2007

Epistemic Foundations of Game Theory

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Assignments for Lecture 2

1. Consider the following "all pay" auction. An envelope containing \$1.75 is auctioned according to the following rules:

1. The bidders are Dave and Melissa. They must take turns bidding, with Dave going first. At each turn, the bidder can pass or bid. The first acceptable bid is 50 cents and each successive bid must exceed the previous bid by exactly 50 cents.

2. The bidding ends once either bidder passes, except at the first bid, where, if the first bidder passes, the second bidder is given the option of bidding herself (and after that the game ends).

3. The highest bidder gets the envelope.

4. **All** bidders must pay the amount of their **last** bid (thus even the loser).

5. The highest bid cannot exceed \$1.00.

(a) Draw the extensive game (bids are public, so it is a game with perfect information).

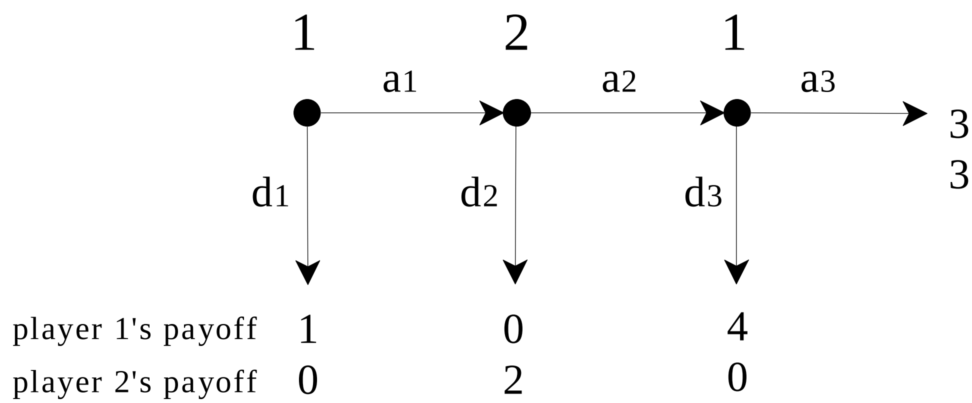
(b) List Dave's strategies and Melissa's strategies.

(c) Write the corresponding strategic-form and find all the pure-strategy Nash equilibria.

(d) Solve the game using backward induction.

(e) Focusing on the strategic-form, what strategy profiles are compatible with common knowledge of rationality? [Use the non-probabilistic notion of rationality from Lecture 1: a player is irrational at a state if and only if there is a strategy of his, different from the one he is actually choosing, that he knows to be better.]

2. Consider the following perfect-information game:



(a) Find the backward-induction solution.

(b) Write the strategic-form corresponding to the extensive game.

(c) Interpret the payoffs as representations of *ordinal* rankings and find the strategy profiles that are consistent with common **knowledge** of rationality. [Use the non-probabilistic notion of rationality from Lecture 1: a player is irrational at a state if and only if there is a strategy of his, different from the one he is actually choosing, that he knows to be better.]

(d) Continue to interpret the payoffs as representations of ordinal rankings and construct an epistemic model of the strategic form of this game satisfying the following properties:

(1) there is a state, say α, where the associated strategy profile yields the play (a_1,a_2,a_3),

(2) at α there is common **knowledge** of rationality.

(e) Now interpret the payoffs as von Neumann Morgenstern payoffs and identify the plays of the game that are consistent with common **knowledge** of rationality. [Use probabilistic notion of rationality from Lecture 1: a player is rational at a state if and only if the strategy he is actually choosing maximizes his expected payoffs, given his beliefs at that state.]

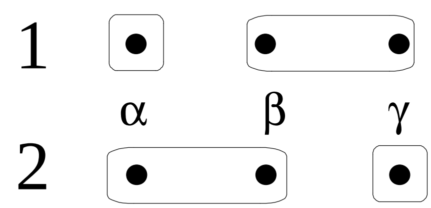
3. Consider again the perfect-information game of Question 2. Let v_1 be the root, v_2 the decision node of player 2 and v_3 the second decision node of player 1.

Given an epistemic model of this game which is based on knowledge (that is, the "doxastic accessibility" relations are equivalence relations), let Ω be the set of states and, for every decision node v_j (j = 1,2,3), let $\|v_j\| \subseteq \Omega$ denote the event that node v_j is reached (clearly, $\|v_1\| = \Omega$). For example, if the strategy profile associated with node α is (a_1a_3, d_2), then $\alpha \in \|v_2\|$ but $\alpha \notin \|v_3\|$.

Let $\mathcal{K}_i$ denote the equivalence relation of player i representing his initial knowledge and, for every $\omega \in \Omega$, let $\mathcal{K}_i(\omega)$ denote the cell of player i's partition that contains state ω. Let $\mathcal{K}_i^{v_j}$ be the equivalence relation of player i representing his knowledge after he has been informed *whether* node v_j has been reached (that is, $\mathcal{K}_i^{v_j}$ represents the initial knowledge $\mathcal{K}_i$ updated by the information that node v_j has been (or has not been) reached). Then $\mathcal{K}_i^{v_j}$ is defined as follows: (1) if $\omega \notin \|v_j\|$ then $\mathcal{K}_i^{v_j}(\omega) = \mathcal{K}_i(\omega) \cap \neg\|v_j\|$ (where $\neg\|v_j\|$ denotes the complement of $\|v_j\|$ in Ω), and (2) if $\omega \in \|v_j\|$ then $\mathcal{K}_i^{v_j}(\omega) = \mathcal{K}_i(\omega) \cap \|v_j\|$.

Denote by V_i the set of decision nodes of player i. Define rationality at a node as follows. At state ω, *player i* is *rational at node v* if and only if either (1) $v \notin V_i$, or (2) $\omega \notin \|v\|$ or (3) $v \in V_i$ and $\omega \in \|v\|$ and player i – when informed that node v has been reached – knows that he would get a higher payoff if he made a different choice at node v than the one that he is actually making. At state ω, player i is *rational at reached nodes* if he is rational at every node.

Consider the following epistemic model of the above game.



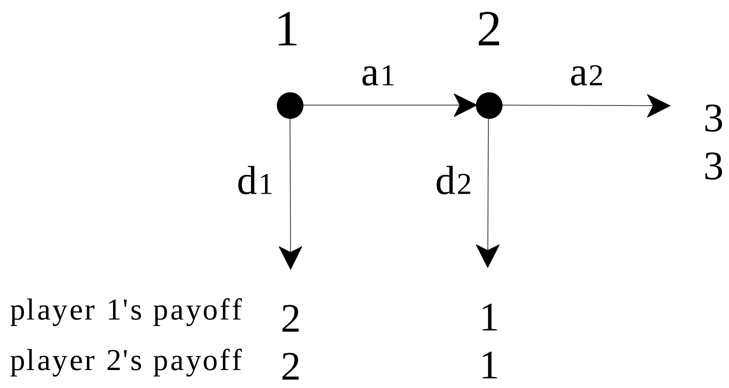
σ_1	a_1d_3	d_1a_3	d_1a_3
σ_2	a_2	a_2	d_2
payoffs	4 0	1 0	1 0

(a) Find the relations $\mathcal{K}_1^{v_2}$, $\mathcal{K}_2^{v_2}$, $\mathcal{K}_1^{v_3}$ and $\mathcal{K}_2^{v_3}$.

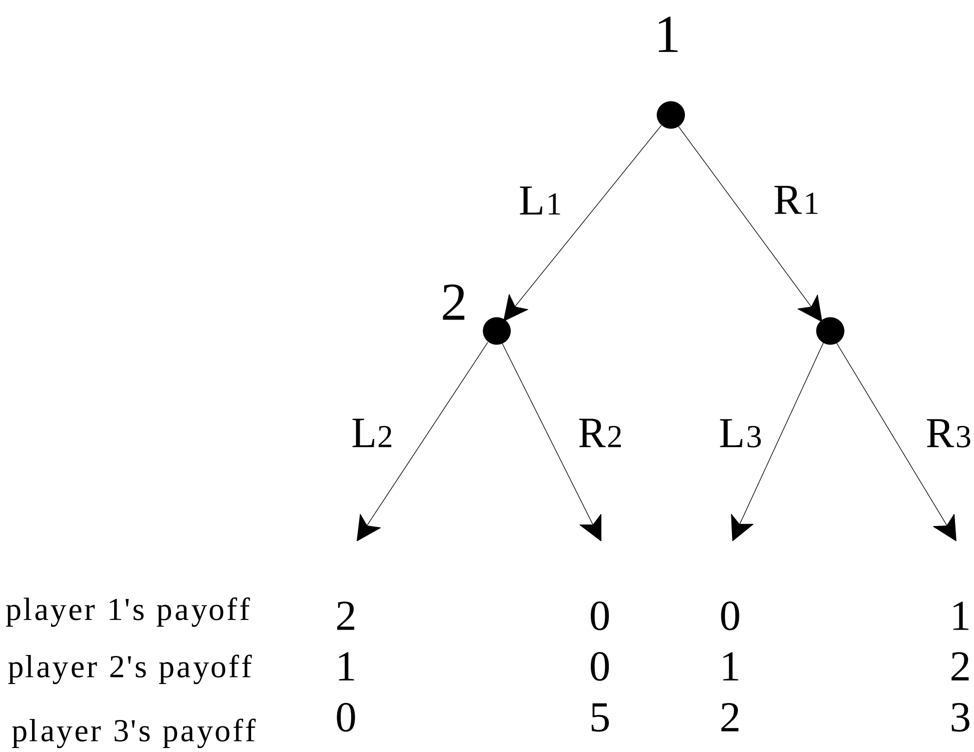
(b) Let $\boldsymbol{R}_i^{v_j}$ denote the event that player i is rational at node v_j. For i =1,2 and j = 1,2,3 determine the events $\boldsymbol{R}_i^{v_j}$.

(c) Let $\boldsymbol{R}_i$ be the event that player i is rational at reached nodes. Determine the events $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $K_1\boldsymbol{R}_2$ (1 knows at the beginning of the game that 2 is rational at reached nodes), $K_2\boldsymbol{R}_1$ and $K_*(\boldsymbol{R}_1 \cap \boldsymbol{R}_2)$ (it is common knowledge at the beginning of the game that both players are rational at reached nodes).

4. Using the notion of rationality at reached nodes, as defined in the previous question, show that in the following game it is consistent with common knowledge or rationality that player 1 terminates the game by playing d_1. That is, construct an epistemic model of the game where there is a state α such that (1) at α player 1 chooses d_1 and (2) at α there is common knowledge of rationality.



5. Consider the following perfect-information game:



(a) Find the backward-induction solution.

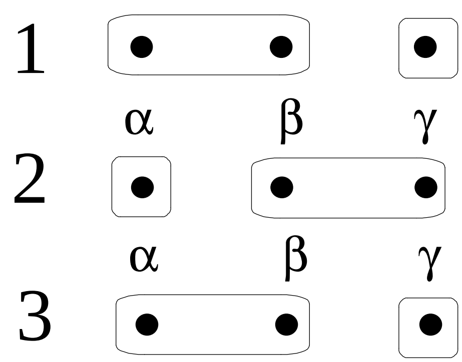
(b) Let v_i be the decision node of player i (i = 1,2,3). For the following epistemic model of this game, find the following relations and events (all the relevant definitions are as in Question 3 above):

(b.1) the relations $\mathcal{K}_i^{v_j}$ for all i =1,2,3 and j = 1,2,3

(b.2) the events $\boldsymbol{R}_i^{v_j}$ for all i =1,2,3 and j = 1,2,3

(b.3) the events $\boldsymbol{R}_i$ for all i =1,2,3

(b.4) the events $(\boldsymbol{R}_1 \cap \boldsymbol{R}_2 \cap \boldsymbol{R}_3)$ and $K_*(\boldsymbol{R}_1 \cap \boldsymbol{R}_2 \cap \boldsymbol{R}_3)$.



σ_1	R_1	R_1	L_1
σ_2	R_2	L_2	L_2
σ_2	R_3	R_3	L_3
payoffs	1 2 3	1 2 3	2 1 0