DEGREE OF MASTER OF SCIENCE
DYNAMIC LOGICS FOR COMMUNICATION
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Problem 1.

1. Each of two children, Alice and Bob, is given a sheet of paper with a natural number written on it. It is common knowledge that the two numbers n_A, n_B belong to the set $\{0, 1, 2, 3\}$, that they are *distinct* ($n_A \neq n_B$) and that *each of the two children can see only the number written on his/her sheet of paper*. We assume the children have *no non-trivial beliefs*: in other words, they believe only what they know. Let us assume that, in fact, Alice's number is 2 and Bob's number is 3. (So $n_A = 2, n_B = 3$.)

   Represent the above situation by a *plausibility model* $(\mathbf{S}, s)$, with two agents (A for Alice, B for Bob) and eight atomic sentences k_A, k_B, with $k \in \{0, 1, 2, 3\}$. The sentence k_A means that Alice's number is k, and similarly for k_B. Clearly specify the set of states, the valuations and the accessibility relations, and point out what is the real state.

2. Alice and Bob are now asked to answer publicly, truthfully and simultaneously the following question: "Is your number even or odd?"

   Assuming that the children do what they are asked, represent this simultaneous announcement as a doxastic action in a (plausibility) action model $\mathbf{\Sigma}$. (Do not use any other atomic sentences but the ones listed above!) Clearly specify the set of actions, the preconditions and the accessibility relations, and point out what is the real action.

3. *Represent (by a plausibility model)* the situation *after* the above announcement, by computing the *anti-lexicographic update product* $\mathbf{S} \otimes \mathbf{\Sigma}$ of the two models. Clearly specify the set of states, the valuations and the accessibility relations, and point out what is the real state.

4. *Does either of the children know now (after the announcement) the other's number? Justify your answer.*

5. (6 marks) If, instead of assuming the numbers to be 2 and 3, we assume that the numbers *have the same parity* (i.e. are both odd or both even), would this make any difference to the answer to the question in the previous part? *Justify your answer, by looking at the possible output-states in this case (after the Alice and Bob truthfully answer the above question).*

6. We now make the same assumption about the numbers as in parts 1 and 2 above (i.e. $n_A = 2, n_B = 3$), but we assume that in fact the two children are *lying* when answering the question: so Alice will say her number is odd, and Bob will say his number is even. It is common knowledge that neither of the children suspects the other is lying: they *both believe (although they don't know) that the other is telling the truth.*

   Represent this "simultaneous lying" announcement, as an action in a plausibility action model $\mathbf{\Sigma}'$.

7. *Represent (as a plausibility state model)* $\mathbf{S}'$ the situation *after the "simultaneous lying" announcement* from the previous part, by computing the *anti-lexicographic update product* $\mathbf{S}' = \mathbf{S} \otimes \mathbf{\Sigma}'$.

8. After the previous (double lying) action, the children are publicly told by their father the following sentence: "*Alice lied*". It is common knowledge that the father always speaks the truth.

   Represent this an action in a plausibility action model $\mathbf{\Sigma}''$.

9. Compute the anti-lexicographic product update $\mathbf{S}' \otimes \mathbf{\Sigma}''$ to represent the situation after father's announcement.

Problem 2. In a far away country, the queen is giving distinctions of honor by placing hats on the heads of remarkable people. Alice and Bob are each due to receive a distinction. It is common knowledge that the queen has only three hats left, namely 2 red hats and 1 white hat. The queen asks Alice and Bob to close their eyes, and then places a hat on each of their heads. Let us suppose that in fact the queen chooses to place *red* hats on *both* their heads. Then they are allowed to look: each can see the other's hat, but not his/her own. Let us assume that both Alice and Bob are cautious players: *at the moment, they only believe what they know (and nothing else).*

1. Represent the above situation as a plausibility model **S**.

2. Now the following action happens: secretely and separately from each other, Alice and Bob look in their mirrors and see their own (*red*!) hats; none of them suspects that the other one is doing this (in other words, they both believe the other is not looking, although they don't know this for sure); and each of them knows that the other doesn't suspect anything.

   But it is common knowledge that the queen can see (and thus knows) anything that happens in her queendom; so e.g. if Bob looks in the mirror, he knows the queen can see him doing this.

   Represent all this scenario as a doxastic action, using a plausibility action model $\mathbf{\Sigma}$, with 4 actions.

3. Let r_b, h_b, r_a, h_a be atomic sentences describing the hat situation (i.e. r_b says that Bob has a red hat) Using the reduction laws (including the derived reduction law for belief), prove that *after the above action, the Queen knows that Alice believes that Bob doesn't know that he has a red hat.* In other words, prove that, if σ is the action described above, the sentence

   $$[\sigma]K_q B_a \neg K_b r_b$$

   is true at the real state in the original model **S**.

Problem 3. Given a discrete conditional-probability space (S, μ), the binary conditional-probability assignments were defined as

$$(s,t)_\mu := \mu(s|\{s,t\})$$

for all $s, t \in S$.

1. Prove that, for any discrete probabilistic space (S, μ), the binary assignments satisfy the following clauses:

$$(s,s) = 1;$$

$$(t,s) = 1 - (s,t), \text{ for } s \neq t \ ;$$

$$(s,w) = \frac{(s,t)\cdot(t,w)}{(s,t)\cdot(t,w) + (w,t)\cdot(t,s)}$$

   for $s \neq t$ and denominator $\neq 0$.

2. Prove the converse of the previous part: given any binary operator $(,) : S \times S \to [0,1]$ on a finite set S, satisfying the above three conditions, there exists a conditional probability measure $\mu : \mathcal{P}(S) \to [0,1]$ such that

$$(s,t) = (s,t)_\mu$$

   for all $s, t \in S$.

Problem 4.

1. Show the *soundness* of the *reduction laws for knowledge and safe belief*: prove that the logical equivalencies claimed by these laws hold at any state in any plausibility model.

2. Prove the *reduction law for belief* (given in the lecture notes) from the reduction laws from knowledge and safe belief. You can also use the other reduction laws and the identity:

$$B_a Q \ = \ \tilde{K}_a \Box_a Q$$

   where $\tilde{K}_a P = \neg K_a \neg P$.

Problem 5 *For advanced students, assumes known standard techniques for proving completeness in Modal Logic*). Briefly sketch the idea for the proof of completeness of the axiomatic system for the *logic of knowledge and safe belief* (as given in the Lecture Notes) *with respect to plausibility models*. Recall these axioms were

- K-axiom for K_a and $\Box_a$;
- $S5$-axioms for K_a;
- $S4$-axioms for $\Box_a$;
- $K_a P \to \Box_a P$;
- $K_a(P \vee \Box_a Q) \wedge (Q \vee \Box_a P) \to K_a P \vee K_a Q$.

Note As usual for normal modalities, we also the following to be part of the axiomatic system: the rules of *modus ponens* and *Necessitation*; and all the propositional tautologies (or, equivalently, any known complete set of axioms for propositional logic).