Completeness and universality for analytic equivalence relation

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Generalized Baire Spaces
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Generalized Descriptive Set Theory is the study of definable subsets of the Generalized Baire Spaces $\kappa^\kappa$, and of all theirs isomorphic spaces.
Basic definitions

Definition

- The $\kappa^+$-Borel sets of a topological space are the ones obtained from the open sets by the operations of complementation and unions of size $\leq \kappa$. 

• A function $f: X \to Y$ is $\kappa^+$-Borel measurable if the preimage of every open $U \subseteq Y$ is $\kappa^+$-Borel.

• $f: X \to Y$ is a $\kappa^+$-Borel isomorphism if $f^{-1}$ exists and is $\kappa^+$-Borel.

• A $\kappa$-space is standard Borel if it is $\kappa^+$-Borel isomorphic to a $\kappa^+$-Borel subset of $\kappa$. 
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▶ A $\kappa$-space is standard Borel if it is $\kappa^+$-Borel isomorphic to a $\kappa^+$-Borel subset of $^\kappa\kappa$.
Analytic sets

Definition
A set $A \subseteq X$ is $\kappa^+$-analytic (or $\Sigma^1_1$) if it is the continuous image of a closed subset of $^{\kappa\kappa}$. 
Definition
Let $X$ and $Y$ be standard Borel $\kappa$-space, and $P, R$ be binary relations over $X$ and $Y$, respectively. We say that $P$ Borel reduces to $R$ (or $P \leq_B R$) if and only if there is a $\kappa^+$-Borel $f : X \to Y$ such that

$$x_1 \ P \ x_2 \iff f(x_1) \ R \ f(x_2).$$
Generalized Borel reducibility

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The notion $\leq_B$ has been used successfully to analyze the complexity of $\Sigma^1_1$ quasi-orders and equivalence relations.
Completeness and universality

Definition
An equivalence relation $E$ on a standard $\kappa^+$-Borel space $X$ is a complete analytic equivalence relation (CAER) if

1. $E \subseteq X^2$ is $\Sigma^1_1$;
2. Every $\Sigma^1_1$ equivalence relation Borel reduces to $E$. 

The classification problem associated to a complete $\Sigma^1_1$ equivalence relation is as complicated as it could be.

While many results in GDST are independent from the model of set theory, a lot of results of completeness are derived from ZFC.
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Some examples

Theorem (Mildenberger-Motto Ros)

The bi-embeddability relation \( \equiv^\kappa_{\text{GRAPHS}} \) is a CAER.

Theorem (C. 2018)

The bi-embeddability relation \( \equiv^\kappa_{\text{TFA}} \) between torsion-free abelian groups of size \( \kappa \) is a CAER.
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The second theorem was derived before establishing the completeness for $\equiv_{\text{TFA}}^\omega = \equiv_{\text{TFA}}^\omega$ in the classical framework.
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Theorem (Mildenberger-Motto Ros)

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*The bi-embeddability relation* \( \equiv^\kappa_{\text{TFA}} \) *between torsion-free abelian groups of size* \( \kappa \) *is a CAER.*

- The second theorem was derived before establishing the completeness for \( \equiv_{\text{TFA}} = \equiv^\omega_{\text{TFA}} \) in the classical framework.
- Now we know that \( \equiv_{\text{TFA}} \) *is a CAER (C.-Thomas), but no explicit reduction from* \( \equiv_{\text{GRAPHS}} \) *to* \( \equiv_{\text{TFA}} \) *is known.*
Proposition
The bi-embeddability relation of $\kappa$-sized structure is a CAER in the following cases.

- Unital rings (ess. Fried, and Sichler 1973);
- Fields (ess. Fried, and Kollár 1982);
- Quandles and others (Brooke-Taylor, and S. Miller);
- ...
Beyond completeness

Let $\mathcal{L}$ be a language of size $\leq \kappa$, and $\varphi \in \mathcal{L}_{\kappa^+}$. 
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Definition

The bi-embeddability relation $\equiv_{\varphi}^\kappa$ is

**invariantly universal** if for every $\Sigma^1_1$ equivalence relation $E$ there is an $\mathcal{L}_{\kappa^+ \kappa}$-sentence $\psi$ such that $X_\psi \subseteq X_\varphi$ and $E \sim_B \equiv \psi$. 
Beyond completeness

Let $\mathcal{L}$ be a language of size $\leq \kappa$, and $\varphi \in \mathcal{L}_{\kappa^+\kappa}$.

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The bi-embeddability relation $\equiv_{\kappa}^\varphi$ is strongly invariant universality if for every $\Sigma^1_1$ equivalence relation $E$ there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence $\psi$ such that $X_\psi \subseteq X_\varphi$ and $E \cong_B \equiv_\psi$. 

$\triangleright$ I.e., there is a bijection between the quotient spaces $f: X/E \to X/\equiv_\psi$ such that both $f$ and $f^{-1}$ admit Borel lifting.
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\[
\begin{array}{ccc}
X & \xrightarrow{\pi_\equiv} & X_\varphi \\
\pi_E & & \pi_\equiv \\
\downarrow & & \downarrow \\
X/E & \xrightarrow{f} & X_\psi/\equiv
\end{array}
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Strong universality

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Theorem (C.-Motto Ros)

The bi-embeddability relation $\equiv^\kappa_{\text{GROUPS}}$ is strongly invariantly universal.

- The methods generalizes for fields, quandles and other structures...
Definition
If $A$ and $B$ are two structures over the languages $\mathcal{K}$ and $\mathcal{L}$, respectively, an interpretation $\Gamma$ of $A$ into $B$ is given by

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such that for all unnested atomic $\mathcal{K}$-formulae $\phi(x_0, \ldots, x_n)$ and all $\bar{b} = (b_0, \ldots, b_n) \in \partial_\Gamma(B)$, we have

$$A \models \phi[f_\Gamma(b_0), \ldots, f_\Gamma(b_n)] \iff B \models \phi_\Gamma[b_0, \ldots, b_n].$$
Let $\mathcal{K} = \{ R \}$ be the language of graphs.
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Lemma (C.-Motto Ros)

There exist a formula $\partial(x)$ and a set of unnested atomic formulæ $\Phi$ in the language of groups such that for each graph $G \in X_{\text{GRAPHS}}$, there is a function $f_G : \partial(H(G)) \rightarrow G$ so that the triple

$$\Gamma := (\partial(x), \Phi, f_G)$$

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Corollary

For every \( \mathcal{K} \)-formula \( \phi(\bar{x}) \) there is a formula \( \phi_\Gamma(\bar{x}) \) in the language of groups such that

\[
G \models \phi[f_G(\bar{a})] \iff H(G) \models \phi_\Gamma[\bar{a}].
\]
Theorem (C.-Motto Ros)

The bi-embeddability relation $\equiv_{\text{GROUPS}}^\kappa$ is strongly invariantly universal.

Sketch.

There is a formula $\varphi_W$ such that if $H \models \varphi_W$ then $H \cong H(G)$, for some graph $G \in X_{\text{GRAPHS}}$.

The map on the quotients $X_{\text{GRAPHS}} / \equiv \to X_{\varphi_W} / \equiv$ induced by $H$ has inverse and admits Borel liftings; and...

...the inverse map has Borel lifting too.
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