Silver dichotomy for countable cofinalities

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Joint work with Xianghui Shi
Previously...
When $\lambda$ is a strong limit cardinal of cofinality $\omega$, descriptive set theory can be done in $\lambda^2$, or equivalently in $\omega\lambda$, $\Pi_{n\in\omega}\lambda_n$ or $V_{\lambda+1}$.

Many results in classical descriptive set theory hold also in this setting.

In general, the results that are dependent to some tree-structure generalize very well.

$I_0(\lambda)$ has an influence on this setting in the same way that AD has an influence on classical descriptive set theory.
Theorem (Silver, 1993)

Let $X$ be a Polish space and $E \subseteq X^2$ be a coanalytic equivalence relation on $X$. Then exactly one of the following holds:

- $E$ has at most countably many classes;
- there is a continuous injection $\varphi : \omega^2 \to X$ such that for distinct $x, y \in \omega^2 \implies \neg \varphi(x)E\varphi(y)$. 

Is this true also for the generalized Baire space?

**Theorem (Friedman, Kulikov 2014)**

Suppose $V = L$ and $\kappa$ inaccessible. Then the order $\langle \mathcal{P}(\kappa), \subset \rangle$ can be embedded into the set of Borel equivalence relations on $2^\kappa$ strictly below the identity, ordered with Borel reducibility.
Theorem (Silver, 1993)

Let $E$ be a coanalytic equivalence relation on $\omega^2$. Then exactly one of the following holds:

- $E$ has at most countably many classes;
- there is a continuous injection $\varphi : 2^\omega \to \omega^2$ such that for distinct $x, y \in 2^\omega \neg \varphi(x)E\varphi(y)$.
Theorem?

Let $E$ be a coanalytic equivalence relation on $\prod_{n \in \omega} \lambda_n$. Then exactly one of the following holds:

- $E$ has at most countably many classes;
- there is a continuous injection $\varphi : \prod_{n \in \omega} \lambda_n \to \prod_{n \in \omega} \lambda_n$ such that for distinct $x, y \in \prod_{n \in \omega} \lambda_n \dashv \varphi(x)E\varphi(y)$. 
Let $E$ be a coanalytic equivalence relation on $\prod_{n\in\omega} \lambda_n$. Then exactly one of the following holds:

- $E$ has at most $\lambda$ many classes;
- there is a continuous injection $\varphi : \prod_{n\in\omega} \lambda_n \rightarrow \prod_{n\in\omega} \lambda_n$ such that for distinct $x, y \in \prod_{n\in\omega} \lambda_n \rightarrow \varphi(x)E\varphi(y)$.
Theorem! (D.-Shi)

Let $\lambda_n$ be measurable cardinals. Let $E$ be a coanalytic equivalence relation on $\prod_{n \in \omega} \lambda_n$. Then exactly one of the following holds:

- $E$ has at most $\lambda$ many classes;
- there is a continuous injection $\varphi : \prod_{n \in \omega} \lambda_n \rightarrow \prod_{n \in \omega} \lambda_n$ such that for distinct $x, y \in \prod_{n \in \omega} \lambda_n$ $\neg \varphi(x)E\varphi(y)$.
“Definition”

Let $E$ be an equivalence relation on some product space. We say that $E$ has the “singleton property” if for all $x, y$, if they differ only in one coordinate, then $\neg xE y$.

Theorem (Shelah 1988)

If $E$ is a co-analytic equivalence relation on $\omega^2$ with the singleton property, then there is a continuous injection $\varphi : \omega^2 \rightarrow \omega^2$ such that for distinct $x, y \in \omega^2$ $\neg \varphi(x)E \varphi(y)$. 
“Definition”

Let $E$ be an equivalence relation on some product space. We say that $E$ has the “singleton property” if for all $x, y$, if they differ only in one coordinate, then $\neg xEy$.

Theorem (Shelah 2003)

Let $\lambda_n$ be measurable cardinals. If $E$ is a co-analytic equivalence relation on $\prod_{n \in \omega} \lambda_n$ with the singleton property, then there is a continuous injection $\varphi : \prod_{n \in \omega} \lambda_n \to \prod_{n \in \omega} \lambda_n$ such that for distinct $x, y \in \prod_{n \in \omega} \lambda_n$ $\neg \varphi(x)E\varphi(y)$. 
Fix a dense subset $S$ of $\langle \omega \rangle^\omega$ that intersects every level in exactly one element. Let $G_0$ be the directed graph that couples two elements if they start with an element of $S$ and differ only in the next coordinate.

**Theorem ($G_0$-dichotomy)**

Let $G$ be an analytic directed graph on $\omega^\omega$. Then exactly one of the following holds:

- there is a (Borel) $\aleph_0$-colouring of $G$;
- there is a continuous function from $\omega^\omega$ to itself that is a homomorphism from $G_0$ to $G$.

This actually generalizes nicely, with almost the same proof.
Fix a dense subset $S$ of $\Pi_{n \in \omega} \lambda_n$ that intersects every level in exactly one element. Let $G_0$ be the directed graph that couples two elements if they start with an element of $S$ and differ only in the next coordinate.

**Theorem?**

Let $G$ be an analytic directed graph on $\Pi_{n \in \omega} \lambda_n$. Then exactly one of the following holds:

- there is a $\aleph_0$-colouring of $G$;
- there is a continuous function from $\Pi_{n \in \omega} \lambda_n$ to itself that is a homomorphism from $G_0$ to $G$.

This actually generalizes nicely, with almost the same proof.
Fix a dense subset $S$ of $\prod_{n \in \omega} \lambda_n$ that intersects every level $n$ in exactly $\kappa_{n-1}$ element. Let $G_0$ be the directed graph that couples two elements if they start with an element of $S$ and differ only in the next coordinate.

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**Theorem! (D.-Shi)**

Let $G$ be an analytic directed graph on $\prod_{n \in \omega} \lambda_n$. Then exactly one of the following holds:

- there is a $\lambda$-colouring of $G$ (actually, something more complicated, but equivalent for graphs that are the complement of an equivalence relation);
- there is a continuous function from $\prod_{n \in \omega} \lambda_n$ to itself that is a homomorphism from $G_0$ to $G$.

This actually generalizes nicely, with almost the same proof.
Now, let $E$ be a co-analytic equivalence relation on $\Pi_{n \in \omega} \lambda_n$. Then its complement $G$ is an analytic directed graph, therefore either $E$ has $\leq \lambda$ equivalence classes, or there is a continuous function $\varphi : \Pi_{n \in \omega} \lambda_n \rightarrow \Pi_{n \in \omega} \lambda_n$ such that $x, y \in G_0$ iff $\neg \varphi(x) E \varphi(y)$. The problem is now that $\varphi$ is possibly not injective.

Classically, from the $G_0$-dichotomy to Silver Dichotomy we use the meagre-comeagre structure of $\omega^2$. This creates many problems in $\Pi_{n \in \omega} \lambda_n$, but Shelah’s theorem can save us: the complement of $G_0$ has the singleton property, and we can use a similar argument to finally prove the Silver Dichotomy.
Can we get rid of the measurable cardinals?

Are measurable cardinals the key to understand the Baire structure of $\lambda^2$?
One of the main points of the Axiom of Determinacy is that it generalizes regularity properties for all subsets of reals. This is true also for Silver Dichotomy:

**Theorem (AD)**

Let $E$ be an equivalence relation on $\omega^2$. Then exactly one of the following holds:

- the classes of $E$ are well-ordered;
- there is a continuous injection $\varphi : \omega^2 \to \omega^2$ such that for distinct $x, y \in \omega^2$ it is not the case that $x E \varphi(y)$. 


One of the main points of I0 is that it generalizes AD-like results to higher cardinal. Does it work also in this case?

**Open problem I0(\lambda)**

Let \( E \) be an equivalence relation on \( \lambda^2 \). Is it true that exactly one of the following holds?

- the classes of \( E \) are well-ordered;

- there is a continuous injection \( \varphi : \lambda^2 \to \lambda^2 \) such that for distinct \( x, y \in \lambda^2 \), \( \neg \varphi(x)E\varphi(y) \).
Forbidden slide 1 (not enough time)

Brief summary of proof of Shelah’s result.
Consider the double diagonal Prikry forcing $\mathbb{P}$ that adds \textit{two} Prikry sequences in $\lambda$. This forcing has two important characteristics:

- if $M$ is a model of cardinality $\lambda$, then there is a $M$-generic set for $\mathbb{P}$ in $V$;
- only the tails of the generic are meaningful, so changing just one coordinate maintain the genericity.
Forbidden slide 2 (not enough time)
The fact that $E$ is co-analytic is also important: this means that the formula that defines $E$ is absolute between models that contain $V_\lambda$.
So the proof goes like this: pick $M$ small model that contains everything. If there is a condition of $\mathbb{P}$ that forces that the two elements of the generic are $E$-related, then also those in $V$ are $E$-related. Switching one coordinate we do the same, but this contradicts the singleton property or the fact that $E$ is an equivalence relation.
Using generic absoluteness, we have a partial result:

**Theorem**

Suppose $\text{I}_0(\lambda)$, as witness by $j$, and let $(\lambda_n)_{n \in \omega}$ be the critical sequence of $j$. Suppose that all subsets of $V_{\lambda+1}$ are $U(j)$-representable. Then if $E \in L(V_{\lambda+1})$ is an equivalence relation with the singleton property, there is a continuous injection $\prod_{n \in \omega} \lambda_n \to \prod_{n \in \omega} \lambda_n$ such that for distinct $x, y \in \prod_{n \in \omega} \lambda_n \neg \varphi(x) E \varphi(y)$. 
Thanks for watching.