Lebesgue’s Density Theorem for Ideals
Part I

David Schrittesser$^1$
joint with Sandra Müller$^1$, Philipp Schlicht$^2$, and Thilo Weinert$^1$

$^1$Kurt Gödel Research Center
University of Vienna

$^2$University of Bristol

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The density points of a set

Let \( \lambda \) denote Lebesgue measure on \( \mathbb{R} \).

**Definition**

Let \( A \subseteq \mathbb{R} \) be measurable and let \( x \in \mathbb{R} \). Define the *density of \( x \) in \( A \) as*

\[
\mathcal{D}_x(A) = \lim_{\varepsilon \to 0} \frac{\lambda(A \cap U_\varepsilon(x))}{\lambda(U_\varepsilon)}
\]

and the set of *points of density 1 in \( A \) or density points of \( A \) as*

\[
\Phi(A) = \{ x \in \mathbb{R} \mid \mathcal{D}_x(A) = 1 \},
\]

- This definition makes sense for any metric space equipped with a Radon measure \( \mu \), such as \( \mathbb{R}^n \) with Lebesgue measure.
- \ldots In particular, for \( \omega \omega \) and \( \omega \mathbb{2} \), each with the usual metric and the usual product measure.
Lebesgue’s Density Theorem

For simplicity, let \((\mathcal{X}, \mu)\) be one of

- \(\mathbb{R}^n\) with the standard metric and Lebesgue measure,
- Cantor space \(\omega^2\) with the usual metric and product measure (the coin-tossing measure).

**Theorem**

*For any \(\mu\)-measurable sets \(A \subseteq \mathcal{X},\)*

\[
\Phi(A) =_{\mu} A.
\]

Note the theorem holds in many more Polish metric measure spaces—
cf. two recent works by Andretta-Camerlo and Andretta-Camerlo-Constantini.
Let $\mathcal{X}$ be a standard Borel space equipped a Borel probability measure $\mu$.

Let $\text{Meas}(\mathcal{X}, \mu)$ denote the measurable, $\text{Null}(\mathcal{X}, \mu)$ the null, and $\text{Borel}(\mathcal{X})$ the Borel subsets of $\mathcal{X}$.

Recall that the measure algebra is defined as

$$\text{Malg} = \frac{\text{Meas}(\mathcal{X}, \mu)}{\text{Null}(\mathcal{X}, \mu)} = \frac{\text{Borel}(\mathcal{X})}{\text{Null}(\mathcal{X}, \mu)}$$

In fact $\text{Malg}$ is always the same, regardless of $\mathcal{X}$ and $\mu$.

Lebesgue’s Density Theorem for $(\mathcal{X}, \mu)$ implies that $\hat{\Phi}$ gives rise to a selector

$$\hat{\Phi} : \text{Malg} \to \text{Meas}(\mathcal{X}, \mu),$$

i.e., $\hat{\Phi}([A]_\mu) \in [A]_\mu$. 

\[ \Phi \text{ as a selector on Malg} \] 

\[ \text{Let } \mathcal{X} \text{ be a standard Borel space equipped a Borel probability measure } \mu. \] 

\[ \text{Let } \text{Meas}(\mathcal{X}, \mu) \text{ denote the measurable, Null}(\mathcal{X}, \mu) \text{ the null, and } \text{Borel}(\mathcal{X}) \text{ the Borel subsets of } \mathcal{X}. \]

\[ \text{Recall that the measure algebra is defined as } \] 

\[ \text{Malg} = \frac{\text{Meas}(\mathcal{X}, \mu)}{\text{Null}(\mathcal{X}, \mu)} = \frac{\text{Borel}(\mathcal{X})}{\text{Null}(\mathcal{X}, \mu)} \]

\[ \text{In fact } \text{Malg} \text{ is always the same, regardless of } \mathcal{X} \text{ and } \mu. \]

\[ \text{Lebesgue’s Density Theorem for } (\mathcal{X}, \mu) \text{ implies that } \hat{\Phi} \text{ gives rise to a selector} \] 

\[ \hat{\Phi} : \text{Malg} \to \text{Meas}(\mathcal{X}, \mu), \]

\[ \text{i.e., } \hat{\Phi}([A]_\mu) \in [A]_\mu. \]
Some properties of $\Phi$

The map $\Phi$ is ‘natural’ in that it’s definition is not too complicated:

- For any measurable set $A$, $\Phi(A)$ is $\Pi_3^0$.
- The following map is the restriction of a $\Sigma_1^1$ relation to a $\Pi_1^1$ set

$$
\Phi \upharpoonright \text{Borel}(\mathcal{X}): \text{Borel}(\mathcal{X}) \to \Pi_3^0
$$

when viewed as a map sending codes to codes.

The map $\Phi$ also has nice algebraic properties, for example:

- $A =_\mu B \Rightarrow \Phi(A) = \Phi(B)$ (well-defined on $[A]_\mu$)
- $A \subseteq_\mu B \Rightarrow \Phi(A) \subseteq \Phi(B)$ (monotonic)
- $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ (preserves $\cap$)
- $\Phi(A) \cap \Phi(A^c) = \emptyset$ (disjointness property, follows from previous)
A Lebesgue Density Theorem for ideals?

From forcing, we know many more ideals, each with their own notion of measurability...

Question

For which of those ideals is there an analogue of Lebesgue’s Density Theorem?

The short answer:

- We can define a notion of density point with nice properties which works for a large class of ccc forcings,
- As a counterexample, we show no even remotely ‘nice’ notion of density point works for Sacks forcing.
Ideals from forcing

Let $\mathfrak{X}$ denote $\omega \omega$ or $\omega 2$, and suppose $\mathbb{P}$ is a set of perfect trees on $\omega$ or $2$, respectively. As usual $[T]$ denotes the set of branches through $T$.

(Of course $\langle \mathbb{P}, \supseteq \rangle$ is a forcing, but we will not force in this talk.)

**Definition**

For $X \subseteq \mathfrak{X}$,

- $X \in N_\mathbb{P} \iff \forall T \in \mathbb{P} \exists T' \in \mathbb{P}$ s.t. $T' \subseteq T$ and $[T'] \cap X = \emptyset$.
- Let $I_\mathbb{P}$ denote the $\sigma$-ideal generated by $I_\mathbb{P}$.
- $X \in I^*_\mathbb{P} \iff \forall T \in \mathbb{P} \exists T' \in \mathbb{P}$ s.t. $T' \subseteq T$ and $[T'] \cap X \in I_\mathbb{P}$.
- $X \in \text{MEAS}_\mathbb{P}(\mathfrak{X}) \iff \forall T \in \mathbb{P} \exists T' \in \mathbb{P}$ s.t. $T' \subseteq T$ and $([T'] \subseteq I^*_\mathbb{P} X$ or $[T'] \subseteq I^*_\mathbb{P} X^C$).

In all cases currently of interest one can show $I_\mathbb{P} = I^*_\mathbb{P}$. 
We make the following assumptions from now on:

- If $T \in \mathbb{P}$ and $s \in T$, $T_s = \{ t \in T \mid t \subseteq s \lor s \subseteq t \} \in \mathbb{P}$.
- $I_P^* = I_P$.
- $\text{Borel}(\mathcal{X}) \subseteq \text{Meas}_P(\mathcal{X})$.

These assumptions hold for a very large class of forcings—e.g., the strongly arboreal forcings that satisfy the ccc or fusion.
Recall $\mathcal{P}$ is a set of perfect trees on $\omega$ or 2. For $T \in \mathcal{P}$, recall that the *stem of $T$* is defined as follows:

$$\text{stem}_T = \max\{t \in T \mid (\forall s \in T) (s \subseteq t \lor t \subseteq s)\}.$$ 

**Definition**

Given $t$ in $\langle \omega \rangle \omega$ or $\langle \omega \rangle 2$ let

$$L_t = \{T \in \mathcal{P} \mid \text{stem}_T = t\}$$

and for $A \in \text{MEAS}_\mathcal{P}(\mathcal{X})$ let

$$\Phi_\mathcal{P}(A) = \{x \in \mathcal{X} \mid (\forall \infty n)(\forall T \in L_{x|n}) [T] \cap A \notin I_\mathcal{P}\}$$

In the relevant case, each set $L_t$ will consist of pairwise compatible conditions...
Example: Random forcing

Let $\mathcal{X} = \omega^2$ with the usual product measure $\mu$ (the coin-tossing measure).

We can regard Random forcing as the set of conditions

$$\mathcal{P} = \{ T \mid T \text{ is a perfect tree on } 2 \text{ and } \frac{\mu([T])}{2\text{lh}(\text{stem}_T)} > \frac{1}{2} \}$$

(ordered by $\supseteq$).

Then for all $A \in \text{MEAS}(\mathcal{X}, \mu)$,

$$\Phi_{\mathcal{P}}(A) = \mu(\Phi(A))$$

(but the two notions of density points don’t coincide).
Equivalently, define

\[ L = L_\emptyset = \{ T \in \mathcal{P} \mid \text{stem}_T = \emptyset \}. \]

**Definition**

- For \( A \subseteq \mathcal{X} \), say \( A \) is \( L \)-positive, or \( A \in L^+ \), iff

  \[ (\forall T \in L) \ [T] \cap A \notin I_P. \]

- Given \( t \) in \( <\omega \omega \) or \( <\omega^2 \), define the shift map \( \sigma_t : \mathcal{X} \to \mathcal{X} \) by

  \[ \sigma_t(x) = t \smallsetminus x \]

- Then

  \[ \Phi_P(A) = \{ x \in \mathcal{X} \mid (\forall \omega n) (\sigma_{x|n})^{-1}[A] \in L^+ \} \]
Suppose $\mathbb{P}$ is strongly linked (cf. the following talk by Sandra Müller), and $\mathbb{P}$ and as well as $\perp_\mathbb{P}$ are $\Sigma^1_1$.

Then $\Phi_\mathbb{P} \upharpoonright \text{Borel}(\mathcal{X})$ is absolutely $\Delta^1_2$ as a map from Borel codes to Borel codes, and for $A, B \in \text{Borel}(\mathcal{X})$

- $\Phi_\mathbb{P}(A) \in [A]_{I_\mathbb{P}}$,  
- $A =_{I_\mathbb{P}} B \Rightarrow \Phi_\mathbb{P}(A) = \Phi_\mathbb{P}(B)$, 
- $\Phi_\mathbb{P}(A) \in \Sigma^0_2$,  
- $A \subseteq_{I_\mathbb{P}} B \Rightarrow \Phi_\mathbb{P}(A) \subseteq_{I_\mathbb{P}} \Phi_\mathbb{P}(B)$ (almost preserves $\subseteq$), 
- $\Phi_\mathbb{P}(A \cap B) =_{I_\mathbb{P}} \Phi_\mathbb{P}(A) \cap \Phi_\mathbb{P}(B)$ (almost preserves $\cap$).

Sandra Müller will also present a theorem that rules out any such map for Sacks forcing satisfying even the first two requirements.
“Does anyone believe that the difference between the Lebesgue and Riemann integrals can have physical significance, and that whether say, an airplane would or would not fly could depend on this difference? If such were claimed, I should not care to fly in that plane.”


Next up: Part II by Sandra Müller
Lebesgue’s Density Theorem for Ideals
Part II

Sandra Müller\textsuperscript{1}
joint with Philipp Schlicht\textsuperscript{2}, David Schrittesser\textsuperscript{1}, and Thilo Weinert\textsuperscript{1}

\textsuperscript{1}Kurt Gödel Research Center
University of Vienna
\textsuperscript{2}University of Bristol

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Density property

Let $\mathcal{I}$ be an ideal and $\Phi : \text{Borel} \to \text{Borel}$ be a function such that for all Borel sets $A$ and $B$, $A =_\mathcal{I} B \Rightarrow \Phi(A) = \Phi(B)$. 

Definition 1

Say that $\Phi$ is $\mathcal{I}$-compatible iff $A \in \mathcal{I} B \Rightarrow A \in \mathcal{I} B$ and $A \setminus B \not\in \mathcal{I}$ for all Borel sets $A$ and $B$.

Definition 2

Say that $\Phi$ is $\mathcal{I}$-positive iff $A \setminus A/2 \not\in \mathcal{I}$ for all Borel sets $A/2$.

Proposition

The following statements are equivalent.

1. $\Phi$ is $\mathcal{I}$-compatible and $\mathcal{I}$-positive.

2. $\Phi$ has the $\mathcal{I}$-density property, i.e. $\Phi(A) = \Phi(B)$ for all Borel sets $A$.

Define the properties we need and state the equivalence. Check that this is really an equivalence, maybe sketch the proof here.
Density property

Let $\mathcal{I}$ be an ideal and $\Phi : \text{Borel} \to \text{Borel}$ be a function such that for all Borel sets $A$ and $B$, $A =_\mathcal{I} B \Rightarrow \Phi(A) = \Phi(B)$.

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   \[ A \subseteq_\mathcal{I} B \Rightarrow \Phi(A) \subseteq_\mathcal{I} \Phi(B) \]

   and

   \[ A \cap B \in \mathcal{I} \Rightarrow \Phi(A) \cap \Phi(B) \in \mathcal{I} \]

   for all Borel sets $A$ and $B$. 

2. Say that $\Phi$ is $\mathcal{I}$-positive iff

   \[ A \setminus A/2 \in \mathcal{I} \Rightarrow \Phi(A) \setminus \Phi(A/2) \in \mathcal{I} \]

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Define the properties we need and state the equivalence. Check that this is really an equivalence, maybe sketch the proof here.
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   for all Borel sets $A$ and $B$.

2. Say that $\Phi$ is $\mathcal{I}$-positive iff $\Phi(A) \cap A \notin \mathcal{I}$ for all Borel sets $A \notin \mathcal{I}$. 
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It is clear that the \( \mathcal{I} \)-density property implies that \( \Phi \) is \( \mathcal{I} \)-positive and \( \mathcal{I} \)-compatible.
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Proof.

It is clear that the \( \mathcal{I} \)-density property implies that \( \Phi \) is \( \mathcal{I} \)-positive and \( \mathcal{I} \)-compatible. For the converse, take any Borel set \( A \). We aim to show that \( \Phi(A) =_\mathcal{I} A \).
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We first show that $B_0 = A \setminus \Phi(A) \in \mathcal{I}$. Towards a contradiction, assume that $B_0 \notin \mathcal{I}$. 

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It is clear that the \( \mathcal{I} \)-density property implies that \( \Phi \) is \( \mathcal{I} \)-positive and \( \mathcal{I} \)-compatible. For the converse, take any Borel set \( A \). We aim to show that \( \Phi(A) =_{\mathcal{I}} A \).

We first show that \( B_0 = A \setminus \Phi(A) \in \mathcal{I} \). Towards a contradiction, assume that \( B_0 \notin \mathcal{I} \). Then \( \Phi(B_0) \setminus \Phi(A) \notin \mathcal{I} \), since it contains \( \Phi(B_0) \cap B_0 \) as a subset, and the latter is not in \( \mathcal{I} \) since \( \Phi \) is \( \mathcal{I} \)-positive.
Proposition

The following statements are equivalent.

1. $\Phi$ is $\mathcal{I}$-compatible and $\mathcal{I}$-positive.

2. $\Phi$ has the $\mathcal{I}$-density property, i.e. $\Phi(A) =_{\mathcal{I}} A$ for all Borel sets $A$.

Proof.

It is clear that the $\mathcal{I}$-density property implies that $\Phi$ is $\mathcal{I}$-positive and $\mathcal{I}$-compatible. For the converse, take any Borel set $A$. We aim to show that $\Phi(A) =_{\mathcal{I}} A$.

We first show that $B_0 = A \setminus \Phi(A) \in \mathcal{I}$. Towards a contradiction, assume that $B_0 \notin \mathcal{I}$. Then $\Phi(B_0) \setminus \Phi(A) \notin \mathcal{I}$, since it contains $\Phi(B_0) \cap B_0$ as a subset, and the latter is not in $\mathcal{I}$ since $\Phi$ is $\mathcal{I}$-positive. On the other hand, we have $\Phi(B_0) \setminus \Phi(A) \in \mathcal{I}$ since $B_0 \subseteq_{\mathcal{I}} A$ and $\Phi$ is $\mathcal{I}$-compatible.
Proposition

The following statements are equivalent.

1. \( \Phi \) is \( \mathcal{I} \)-compatible and \( \mathcal{I} \)-positive.
2. \( \Phi \) has the \( \mathcal{I} \)-density property, i.e. \( \Phi(A) =_{\mathcal{I}} A \) for all Borel sets \( A \).

Proof.

It remains to show that \( B_1 = \Phi(A) \setminus A \in \mathcal{I} \).
Proposition

The following statements are equivalent.

1. \( \Phi \) is \( \mathcal{I} \)-compatible and \( \mathcal{I} \)-positive.
2. \( \Phi \) has the \( \mathcal{I} \)-density property, i.e. \( \Phi(A) = \mathcal{I} A \) for all Borel sets \( A \).

Proof.

It remains to show that \( B_1 = \Phi(A) \setminus A \in \mathcal{I} \). Assume that \( B_1 \notin \mathcal{I} \), so in particular \( \Phi(A) \notin \mathcal{I} \). The set \( C = \Phi(B_1) \cap B_1 \notin \mathcal{I} \), since \( \Phi \) is \( \mathcal{I} \)-positive.
Proposition

The following statements are equivalent.

1. Φ is $\mathcal{I}$-compatible and $\mathcal{I}$-positive.
2. Φ has the $\mathcal{I}$-density property, i.e. $\Phi(A) =_\mathcal{I} A$ for all Borel sets A.

Proof.

It remains to show that $B_1 = \Phi(A) \setminus A \in \mathcal{I}$. Assume that $B_1 \notin \mathcal{I}$, so in particular $\Phi(A) \notin \mathcal{I}$. The set $C = \Phi(B_1) \cap B_1 \notin \mathcal{I}$, since Φ is $\mathcal{I}$-positive. We have $\Phi(B_1) \cap \Phi(A) \in \mathcal{I}$ since $B_1 \cap A = \emptyset$ and Φ is $\mathcal{I}$-compatible. Hence $C \subseteq \Phi(B_1)$ implies $C \cap \Phi(A) \in \mathcal{I}$. 
Proposition

The following statements are equivalent.
1. \( \Phi \) is \( \mathcal{I} \)-compatible and \( \mathcal{I} \)-positive.
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Proof.

It remains to show that \( B_1 = \Phi(A) \setminus A \in \mathcal{I} \). Assume that \( B_1 \notin \mathcal{I} \), so in particular \( \Phi(A) \notin \mathcal{I} \). The set \( C = \Phi(B_1) \cap B_1 \notin \mathcal{I} \), since \( \Phi \) is \( \mathcal{I} \)-positive. We have \( \Phi(B_1) \cap \Phi(A) \in \mathcal{I} \) since \( B_1 \cap A = \emptyset \) and \( \Phi \) is \( \mathcal{I} \)-compatible. Hence \( C \subseteq \Phi(B_1) \) implies \( C \cap \Phi(A) \in \mathcal{I} \). However, this contradicts the fact that \( C \subseteq_{\mathcal{I}} B_1 \subseteq_{\mathcal{I}} \Phi(A) \). \( \square \)
For which ideals $\mathcal{I}$ is our density function $\Phi_s$ $\mathcal{I}$-compatible?
For which ideals $\mathcal{I}$ is our density function $\Phi_s$ $\mathcal{I}$-compatible?

**Definition**

A tree forcing $\mathbb{P}$ has the *stem property* if for all $T \in \mathbb{P}$ and $\mathcal{I}$-almost all $x \in [T]$, there are infinitely many $n \in \omega$ such that there is some $T' \leq T$ with $x \in [T']$ and $\text{stem}_{T'} = x \upharpoonright n$. 

**Lemma**

Let $\mathbb{P}$ be a ccc tree forcing with the stem property and $\mathcal{I} = \mathcal{I}_\mathbb{P}$. Then $s$ is $\mathcal{I}$-compatible.
For which ideals $\mathcal{I}$ is our density function $\Phi_s$ $\mathcal{I}$-compatible?

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A tree forcing $\mathbb{P}$ has the *stem property* if for all $T \in \mathbb{P}$ and $\mathcal{I}$-almost all $x \in [T]$, there are infinitely many $n \in \omega$ such that there is some $T' \leq T$ with $x \in [T']$ and $\text{stem}_{T'} = x \upharpoonright n$.

**Lemma**

*Let $\mathbb{P}$ be a ccc tree forcing with the stem property and $\mathcal{I} = \mathcal{I}_\mathbb{P}$. Then $\Phi_s$ is $\mathcal{I}$-compatible.*
$\mathcal{I}$-positivity for strongly linked forcings

For which ideals $\mathcal{I}$ is our density function $\Phi_s$ $\mathcal{I}$-positive?
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**Definition**

A tree forcing $\mathbb{P}$ is *strongly linked* if any $S, T \in \mathbb{P}$ with $\text{stem}_S \subseteq \text{stem}_T$ and $\text{stem}_T \in S$ are compatible in $\mathbb{P}$. 

Note that strongly linked implies $\mathcal{I}$-linked and hence ccc.

**Lemma**

Let $\mathbb{P}$ be a strongly linked tree forcing with the stem property and $\mathcal{I} = \mathcal{I}_\mathbb{P}$. Let $T \in \mathbb{P}$. Then $\mathcal{I}$-almost all $x \in [T]$ are $\mathcal{I}$-density points of $[T]$. This implies that $\Phi_s$ is $\mathcal{I}_\mathbb{P}$-positive.
For which ideals $\mathcal{I}$ is our density function $\Phi_s \mathcal{I}$-positive?

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A tree forcing $\mathcal{P}$ is *strongly linked* if any $S, T \in \mathcal{P}$ with $\text{stem}_S \subseteq \text{stem}_T$ and $\text{stem}_T \in S$ are compatible in $\mathcal{P}$.

Note that strongly linked implies $\sigma$-linked and hence ccc.
\( \mathcal{I} \)-positivity for strongly linked forcings

For which ideals \( \mathcal{I} \) is our density function \( \Phi_s \) \( \mathcal{I} \)-positive?

**Definition**

A tree forcing \( \mathbb{P} \) is **strongly linked** if any \( S, T \in \mathbb{P} \) with \( \text{stem}_S \subseteq \text{stem}_T \) and \( \text{stem}_T \in S \) are compatible in \( \mathbb{P} \).

Note that strongly linked implies \( \sigma \)-linked and hence ccc.

**Lemma**

*Let \( \mathbb{P} \) be a strongly linked tree forcing with the stem property and \( \mathcal{I} = \mathcal{I}_\mathbb{P} \). Let \( T \in \mathbb{P} \). Then \( \mathcal{I} \)-almost all \( x \in [T] \) are \( \mathcal{I} \)-density points of \( [T] \).*

This implies that \( \Phi_s \) is \( \mathcal{I}_\mathbb{P} \)-positive.
Corollary

Suppose $\mathcal{P}$ is a strongly linked tree forcing with the stem property and let $\mathcal{I} = \mathcal{I}_P$. Then $\Phi_s$ has the density property.
Suppose $\mathbb{P}$ is a strongly linked tree forcing with the stem property and let $\mathcal{I} = \mathcal{I}_\mathbb{P}$. Then $\Phi_s$ has the density property.

In particular, $\Phi_s$ has the density property for

- Cohen forcing $\mathbb{C}$,
- Hechler forcing $\mathbb{H}$,
- eventually different reals forcing $\mathbb{E}$,
- Laver forcing with a filter $\mathbb{L}_F$, and
- Mathias forcing with a translation invariant filter $\mathbb{R}_F$. 
Ideals without the density property

How about non-ccc ideals?
Ideals without the density property

How about non-ccc ideals?

**Proposition**

\( \Phi_s \) does not have the density property for

- Mathias forcing \( R \),
- Silver forcing \( V \),
- Sacks forcing \( S \),
- Laver forcing \( L \), and
- Miller forcing \( M \).
A strong failure of the density property for Sacks forcing

There is no Baire measurable function $\Phi$ with the density property yielding a notion of density points for the ideal $\mathcal{I}$ of countable sets.
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**Definition**

A *selector* for an equivalence relation is a function that picks an element from each equivalence class. Here we will have equivalence relations $E \subseteq F$ on a set $Y$ and a selector for the equivalence relation induced by $F$ on $Y/E$. 
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**Definition**

A *selector* for an equivalence relation is a function that picks an element from each equivalence class. Here we will have equivalence relations $E \subseteq F$ on a set $Y$ and a selector for the equivalence relation induced by $F$ on $Y/E$.

Let $\Lambda$ denote the set of Borel codes and let $B_x$ denote the set with code $x \in \Lambda$. Moreover, consider the following equivalence relations on $\Lambda$:

$$(x, y) \in E_\Xi \iff B_x = B_y$$

and

$$(x, y) \in E_\mathcal{I} \iff B_x \triangle B_y \in \mathcal{I}.$$
A strong failure of the density property for Sacks forcing

**Definition**

A selector for $\mathcal{I}$ with Borel values is a selector for $E_\mathcal{I}/E_=$ on $\Lambda$.

Theorem

There is no Baire measurable selector for $\mathcal{I}$ with Borel values.

Almost the same proof also shows:

Theorem 1

There is no Baire measurable selector for $\mathcal{I}$ with $\mathcal{I}_2$ values.

Assuming PD, there is no Baire measurable selector for $\mathcal{I}$ with projective values.
A strong failure of the density property for Sacks forcing

Definition

A selector for $\mathcal{I}$ with Borel values is a selector for $E_\mathcal{I}/E_=$ on $\Lambda$.

Theorem

There is no Baire measurable selector for $\mathcal{I}$ with Borel values.

Almost the same proof also shows:

Theorem 1

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A strong failure of the density property for Sacks forcing

**Definition**

A selector for $\mathcal{I}$ with Borel values is a selector for $E_\mathcal{I}/E_=$ on $\Lambda$.

**Theorem**

There is no Baire measurable selector for $\mathcal{I}$ with Borel values.

Almost the same proof also shows:

**Theorem**

1. There is no Baire measurable selector for $\mathcal{I}$ with $\Sigma^1_2$ values.
2. Assuming PD, there is no Baire measurable selector for $\mathcal{I}$ with projective values.
Open questions

**Question**

*Is the existence of a simply definable selector equivalent to the ccc for all homogeneous $\sigma$-ideals?*
Open questions

Question

Is the existence of a simply definable selector equivalent to the ccc for all homogeneous $\sigma$-ideals?

Question

Is there a Baire measurable selector with Borel values for other non-ccc ideals?
“This new integral of Lebesgue is proving itself a wonderful tool. I might compare it with a modern Krupp gun, so easily does it penetrate barriers which were impregnable.”


Thank you for your attention!