The tree property at $\aleph_{\omega+2}$

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Recall that an uncountable regular cardinal \( \kappa \) has the tree property (TP(\( \kappa \))) if every \( \kappa \)-tree has a cofinal branch.

In this talk we show the basic steps behind the proof of the following theorem:

**Theorem (Friedman, Honzík, S. (2018))**

\((GCH)\) Suppose \( 0 \leq n < \omega \) is a natural number and there is \( \kappa \) which is \( H(\lambda^{+n}) \)-hypermeasurable where \( \lambda \) is the least weakly compact above \( \kappa \), then there is a forcing extension where the following hold:

1. \( \kappa = \aleph_\omega \) is strong limit and \( 2^{\aleph_\omega} = \aleph_{\omega+n+2} \).
2. TP(\( \aleph_{\omega+2} \)).
Recall that if $\aleph_\omega$ is strong limit, then by a result by Shelah,

$$2^{\aleph_\omega} < \min(\aleph_{2\omega+}, \aleph_{\omega_4}),$$

so we cannot aim for an arbitrary infinite gap.

We will mention at the end some open question, in particular whether we can extend our result to a countable gap with TP$(\aleph_{\omega+2})$. 
Forcing we will use

We use Mitchell forcing because it is more suitable to manipulate the continuum function.

- The product of the Mitchell and the Cohen forcing works nicely because Mitchell projects to the Cohen forcing (at relevant cardinals).
- The Mitchell forcing $\mathbb{M}(\kappa, \lambda)$ can be easily modify to force $2^\kappa > \kappa^{++}$ while forcing $\text{TP}(\kappa^{++})$. 
Assume $\kappa < \lambda$ are infinite regular cardinals, with $\lambda$ being inaccessible (weakly compact for us).

**Definition**

A condition in $\mathbb{M}(\kappa, \lambda)$ is a pair $(p, q)$ such that $p$ is a condition in $\text{Add}(\kappa, \lambda)$ and $q$ is a function with domain of size at most $\kappa$, $\text{Dom}(q) \subseteq \lambda$, such that for all $\alpha \in \text{Dom}(q)$, $q(\alpha)$ is an $\text{Add}(\kappa, \alpha)$-name for a condition in $\text{Add}(\kappa^+, 1)^{\text{Add}(\kappa, \alpha)}$.

The ordering is defined on the next slide.
Assume $\kappa < \lambda$ are infinite regular cardinals, with $\lambda$ being inaccessible. Then

**Definition**

A condition $(p, q)$ is stronger than $(p', q')$ if

(i) $p \leq p'$,

(ii) $\text{dom}(q) \supseteq \text{dom}(q')$ and for every $\beta \in \text{dom}(q')$,

\[ p \upharpoonright \beta \models \text{Add}(\kappa, \beta) \quad q(\beta) \leq q'(\beta). \]
Mitchell forcing, basic properties

Assuming that $\kappa^\kappa = \kappa$ and $\lambda > \kappa$ is an inaccessible cardinal, Mitchell forcing $M(\kappa, \lambda)$ satisfies following:

- It is $\lambda$-Knaster and $\kappa$-closed.
- It collapses the cardinals in the open interval $(\kappa^+, \lambda)$ to $\kappa^+$.
- It forces $2^\kappa = \lambda = \kappa^{++}$.

There is a projection from $M(\kappa, \lambda)$ to $\text{Add}(\kappa, \lambda)$.

The preservation of $\kappa^+$ is shown by the existence of a projection from $\text{Add}(\kappa, \lambda) \times T$ to $M(\kappa, \lambda)$, where $T$ is a $\kappa^+$-closed forcing (it has conditions of the form $(0, q)$ in $M(\kappa, \lambda)$).\(^1\)

The natural projection from $M(\kappa, \lambda)$ to $M(\kappa, \alpha)$ for $\kappa < \alpha < \lambda$, makes it possible to treat $M(\kappa, \lambda)$ as an iteration, and write $M(\kappa, \alpha) * \dot{R}$.

\(^1\)We call $T$ the term forcing.
Branch lemmas

Let $\kappa$, $\lambda$ be regular cardinals.

- (essentially Baumgartner) Assume that $\mathbb{P} \times \mathbb{P}$ is a $\kappa$-cc forcing notion. If $T$ is a tree of height $\kappa$, then forcing with $\mathbb{P}$ does not add cofinal branches to $T$.

- (essentially Silver) Let $\kappa < \lambda$, with $2^\kappa \geq \lambda$. Assume that $\mathbb{P}$ is a $\kappa^+$-closed forcing notion. If $T$ is a $\lambda$-tree, then forcing with $\mathbb{P}$ does not add cofinal branches to $T$. 
The main strategy of the proof, with gap 3 (n=1)

- We prepare the universe $V$ so that forcing $2^\kappa = \lambda^+$ with the Cohen forcing will preserve the measurability of $\kappa$ (with some work, this is possible to do with the large-cardinal assumption that $\kappa$ has a $(\kappa, \lambda^+)$-extender; supercompactness is not necessary$^2$).
  - The preparation actually destroys the strong-limitness of $\lambda$. Thus $\lambda$ is not weakly compact in the rest of the argument. This presents a technical obstacle which needs to be overcome.
- We use a variant of the Mitchell forcing $\mathbb{M} = \mathbb{M}(\kappa, \lambda, \lambda^+)$ to force $2^\kappa$ to be equal to $\lambda^+$, and simultaneously collapse cardinals in the interval $(\kappa, \lambda)$.

In $V[\mathbb{M}]$, the tree property holds at $\kappa^{++}$, $\kappa$ is still measurable, and we can define a Prikry forcing with collapses. Our final forcing is

$$\mathbb{M}(\kappa, \lambda, \lambda^+) \ast \mathbb{Q},$$

where $\mathbb{Q}$ is the Prikry forcing with collapses (defined with respect to some guiding generic).

Now, the quotient analysis is much harder because of the Prikry forcing (with collapses).

Let give a brief review of the quotient analysis on the next slide.
Let \( k : V \rightarrow M \) be an elementary embedding with critical point \( \lambda \). With the right setup we can write

\[
k(M \ast Q) = (M \ast Q) \ast R,
\]

where \( R \) is the quotient forcing \( k(M \ast Q)/(M \ast Q) \). In particular, if \( G \ast x \) is \( M \ast Q \)-generic, then

\[
R = \{(p', q', r') \in k(M \ast Q) \mid (p', q', r')||k''(G \ast x)\}.
\]

We wish to show that \( R \) does not add branches to \( \lambda \)-trees over \( M[G][x] \).
Unlike the classical case (just with $\mathbb{M}$), it is not clear whether $\mathbb{R}$ regularly embeds into a product $P_1 \times P_2$, where $P_1 \times P_2$ is $\kappa^+$-cc and $P_2$ is $\kappa^+$-closed, which would make the argument simpler.

Instead we will show directly that $\mathbb{R}$ does not add branches, which requires a careful analysis of when $(p, q, r)$ in $\mathbb{M} \ast \mathbb{Q}$ forces $(p', q', r')$ in $k(\mathbb{M} \ast \mathbb{Q})$ into (or out of) $\dot{\mathbb{R}}$.

A rough outline of the argument is given on next slide.
Suppose for contradiction $T$ is a $\lambda$-tree in $V[M \ast Q]$ and $R$ forces that $\dot{b}$ is a new branch in $T$. Let $\dot{T}$ be a $Q$-name over $M$ for $T$.

We build a labelled tree $\mathcal{T}$ of height $\kappa$ of conditions $a = (r, (p', q', r'))$ in $Q \ast R$ such that $r$ decides how $\dot{T}$ looks locally and the whole $a$ decide how $\dot{b}$ looks locally. In particular if $(r, (p', q', r'))$ decides a segment of $\dot{b}$ through $\dot{T}$, and for instance knows $y < \dot{T} z$ are in $\dot{b}$, then already $r$ knows $y < \dot{T} z$.

Since $\mathcal{T}$ has $2^\kappa$ cofinal branches, there are two branches $v, w$ through $\mathcal{T}$ and respective conditions $a_v$ and $a_w$ which decide $\dot{b}|\delta$ the same way, say $y$ (where $\delta$ is a level of $\dot{T}$ such that $\dot{b}|\delta$ is being decided by branches through $\mathcal{T}$).

Continuing above these conditions, we get two more conditions which decide a restriction of $\dot{b}$ above $y$ differently.
• By reflection (which is built into $\mathcal{T}$), such a difference is by necessity reflected down to some level $\delta' < \delta$ which contradicts the fact that $a, b$ decide the restriction $b|\delta = y$ the same way.

• The construction of $\mathcal{T}$ and the subsequent arguments use crucially the fact that we work with a dense subforcing of $Q \ast R$ in which the conditions $(r, (p', q', r'))$ are such that $r \in Q$ and $r' \in k(Q)$ have the same stem.

• With conditions from this dense subforcing, one can extend $p'$ and $q'$ more easily without running the risk of incompatibility with the stem of $r$ (which would result in falling out of the quotient $R$). Then we use the nice chain condition of "$p$"-conditions and nice closure of the "$q$"-conditions (with respect to the term ordering) to build $\mathcal{T}$.
A variant of this argument is used to show any finite gap with $\text{TP}(\aleph_{\omega+2})$. In this variant, we essentially reduce the general case to the gap 3 case.
Open questions:

1. Is it consistent to have an infinite gap with $\text{TP}(\aleph_{\omega+2})$?
2. Can we in addition control other cardinal invariants besides $c(\aleph_{\omega})$? For instance $c(\aleph_{n})$ for $n < \omega$, or $u(\aleph_{\omega})$?