Kurepa trees and spectra of $\mathcal{L}_{\omega_1,\omega}$ sentences

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Consistency results involving Kurepa trees.

Application: analyzing the spectrum of an $L_{\omega_1,\omega}$ sentence.
Let $\phi$ be an an $\mathcal{L}_{\omega_1,\omega}$ sentence. The **spectrum** of $\phi$ is the set of all cardinalities of models of $\phi$ i.e.

$$\text{Spec}(\phi) = \{\kappa \mid \exists M \models \phi, |M| = \kappa\}$$

If $\text{Spec}(\phi) = [\aleph_0, \kappa]$, then $\phi$ **characterizes** $\kappa$.

General question: which cardinals can be characterized?

Some facts:

- (Morley, Lopez-Escobar) Let $\Gamma$ be a countable set of $\mathcal{L}_{\omega_1,\omega}$ sentences. If $\Gamma$ has models of cardinality $\aleph_\alpha$ for all $\alpha < \omega_1$, then it has models in all infinite cardinalities.

- (Hjorth, 2002) For all $\alpha < \omega_1$, $\aleph_\alpha$ is characterized by a complete $\mathcal{L}_{\omega_1,\omega}$ sentence.

Corollary: Under GCH, $\aleph_\alpha$ is characterized by a complete $\mathcal{L}_{\omega_1,\omega}$ sentence iff $\alpha < \omega_1$. 
Motivation

Corollary

*Under GCH, $\aleph_\alpha$ is characterized by a complete $L_{\omega_1, \omega}$ sentence iff $\alpha < \omega_1$.***

**Question:**

Can there exist an $L_{\omega_1, \omega}$ sentence that characterizes $\aleph_{\omega_1}$? (Under failure of GCH)

**Answer:** Yes.

A conjecture of Shelah’s: If $\aleph_{\omega_1} < 2^{\aleph_0}$, then any $L_{\omega_1, \omega}$ sentence which has models of size $\aleph_{\omega_1}$ also has models of size $2^{\aleph_0}$.

We show: $2^{\aleph_0}$ cannot be replaced by $2^{\aleph_1}$ in the above.
The model theoretic application

We show the following:
There exists an $\mathcal{L}_{\omega_1,\omega}$ sentence $\phi$, for which it is consistent with ZFC that:

1. $\phi$ characterizes $\aleph_{\omega_1}$, i.e. it has spectrum $[\aleph_0, \aleph_{\omega_1}]$.
2. $2^{\aleph_0} < \aleph_{\omega_1} < 2^{\aleph_1}$ and $\phi$ has models of size $\aleph_{\omega_1}$, but not $2^{\aleph_1}$.
3. The spectrum of $\phi$ can be $[\aleph_0, 2^{\aleph_1})$ where $2^{\aleph_1}$ is weakly inaccessible.

Note: this is the first example where the spectrum of a sentence can be both right-open and right-closed.

We define $\phi$ to code a Kurepa tree.
Definition

$T$ is a **Kurepa tree** if $T$ has countable levels, height $\aleph_1$, and at least $\aleph_2$ many cofinal branches.

For $\lambda > \omega_1$, $KH(\aleph_1, \lambda)$ is the statement that there exists a Kurepa tree with $\lambda$ many branches.

$$B := \sup \{ \lambda \mid KH(\aleph_1, \lambda) \text{ holds} \}$$

Note that $\aleph_2 \leq B \leq 2^{\aleph_1}$

Similarly, for any regular $\kappa$, can define $\kappa$-Kurepa trees, $KH(\kappa, \lambda)$ and $B(\kappa)$, where $\kappa$ is the height of the tree in place of $\aleph_1$; $\kappa^+ \leq B(\kappa) \leq 2^\kappa$. 

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**Theorem**

*There is an $\mathcal{L}_{\omega_1,\omega}$ sentence $\phi$, such that $\phi$ has a model of size $\lambda$ iff $\lambda \leq 2^{\aleph_0}$ or there is a Kurepa tree with $\lambda$ many branches (i.e. $KH(\omega_1, \lambda)$).*

In other words,

- If there are no Kurepa trees, $\text{Spec}(\phi) = [\aleph_0, 2^{\aleph_0}]$;
- If $B$ is a maximum, then $\phi$ characterizes $\max(2^{\aleph_0}, B)$.
Consistency results

\[ \mathcal{B} := \sup \{ \lambda \mid KH(\omega_1, \lambda) \text{ holds} \} \]

**Theorem**

*It is consistent with ZFC, that:*

1. \(2^{\aleph_0} < \aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_1}\) and there exist a Kurepa tree with \(\aleph_{\omega_1}\) many branches.

2. \(\aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_0}\) and there exist a Kurepa tree with \(\aleph_{\omega_1}\) many branches.

*Note that in both cases \(\mathcal{B}\) is a maximum.*

**The model theoretic application:**

**Corollary**

*There is a \(L_{\omega_1, \omega}\) sentence \(\phi\), which consistently:*

- characterizes \(2^{\aleph_0}\),
- characterizes \(\aleph_{\omega_1}\) and \(2^{\aleph_0} < \aleph_{\omega_1}\).*
An overview of the proof

Theorem

It is consistent with ZFC, that:

1. \(2^{\aleph_0} < \aleph_{\omega_1} = B < 2^{\aleph_1}\) and there exist a Kurepa tree with \(\aleph_{\omega_1}\) many branches.

2. \(\aleph_{\omega_1} = B < 2^{\aleph_0}\) and there exist a Kurepa tree with \(\aleph_{\omega_1}\) many branches.

Let \(V \models ZFC + GCH\).

The forcing posets:

- Let \(\mathbb{P}\) be the standard \(\sigma\)-closed, \(\aleph_2\)-c.c. poset to add a Kurepa tree with \(\aleph_{\omega_1}\) many branches.

- Let \(\mathbb{C} := \text{Add}(\omega, \aleph_{\omega_1+1})\)

Then, we claim that

1. \(V[\mathbb{P}]\) gives part (1)

2. \(V[\mathbb{P} \times \mathbb{C}]\) gives part (2).
An overview of the proof

Some key points in the proof of (2):

- $\mathbb{P}$ adds a Kurepa tree with $\aleph_{\omega_1}$-many branches, showing that $\mathcal{B} \geq \aleph_{\omega_1}$.

- For $\alpha < \omega_1$, let $\mathbb{P}_\alpha$ be the restriction of $\mathbb{P}$ that adds the first $\aleph_\alpha$ many branches to the generic tree.

$\mathcal{B} \leq \aleph_{\omega_1}$:

- Let $T$ be a Kurepa tree in $V[\mathbb{P}][C]$.
- Then $T \in V[\mathbb{P}][Add(\omega, \omega_1)]$, for an appropriately chosen generic $Add(\omega, \omega_1)$.
- Every cofinal branch of $T$ is in $V[\mathbb{P}_\alpha][Add(\omega, \omega_1)]$, for some $\alpha < \omega_1$.
- In $V[\mathbb{P}_\alpha][Add(\omega, \omega_1)]$, $2^{\omega_1} < \aleph_{\omega_1}$.

Then, by cardinal arithmetic, $T$ cannot have more that $\aleph_{\omega_1}$ many branches.

Corollary: The sentence $\phi$ can characterize $\aleph_{\omega_1}$. 
In the above theorem, we force $B$ to be a maximum. And in part (1), $\text{Spec}(\phi) = [\aleph_0, \aleph_{\omega_1}]$.

**Question:** Can we have $B$ be a supremum, but not a maximum? More generally, can the spectrum of an $\mathcal{L}_{\omega_1,\omega}$ sentence consistently be both right-hand closed and open?

It turns out, yes.

From a Mahlo cardinals, we force $B = 2^{\aleph_1}$ and no Kurepa trees with $2^{\aleph_1}$ many branches.
$B$ can be a supremum, not a maximum:

**Theorem**

*From a Mahlo cardinal, it is consistent that $2^{\aleph_0} < B = 2^{\aleph_1}$, for every $\kappa < 2^{\aleph_1}$, there is a Kurepa tree with at least $\kappa$ many branches, but there is no Kurepa tree with $2^{\aleph_1}$ many branches.*

Key notions in the proof:

- The forcing axiom GMA;
- A maximality principle, SMP;
- Their consequences on $\Sigma^1_1$ subsets of $\omega^{\omega_1}$.
A forcing axiom, defined by Shelah.

Some definitions:

Let $\kappa$ be regular; a poset is **stationary $\kappa^+$-linked** if for every sequence $\langle p_\gamma | \gamma < \kappa^+ \rangle$, there is a regressive $f : \kappa^+ \to \kappa^+$, s.t. for almost all $\gamma, \delta \in \kappa^+ \cap \text{cof}(\kappa)$, $f(\gamma) = f(\delta)$ implies that $p_\gamma, p_\delta$ are compatible.

Set $\Gamma_{\kappa}$ to be the collection of all $\kappa$-closed, stationary $\kappa^+$-linked, well met posets with greatest lower bounds.

**Definition**

$GMA_{\kappa}$ states that every $\mathbb{P} \in \Gamma_{\kappa}$ for every collection of dense sets $\mathcal{D} \subset \mathbb{P}$ with $|\mathcal{D}| < 2^\kappa$, there exists a $\mathcal{D}$-generic filter for $\mathbb{P}$. 
A maximality principle, that generalizes GMA.

**Definition**
For a regular $\kappa$, $SMP_n(\kappa)$ states that:

- $\kappa^{<\kappa} = \kappa$;
- for any $\Sigma_n$ formula $\phi$, with parameters in $H(2^\kappa)$ and any $\mathbb{P} \in \Gamma_\kappa$, if for all $\kappa$-closed, $\kappa^+$-c.c. $Q \in V[\mathbb{P}]$, $V[\mathbb{P}][Q] \models \phi$, then $\phi$ is true in $V$.

$SMP_\kappa$ means $SMP_n(\kappa)$ for all $n$.

**Fact** (Philipp Lücke): If $\kappa^{<\kappa} = \kappa$ and there is a Mahlo $\theta > \kappa$, then one can force $SMP(\kappa)$. 

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Some implications

Proposition

(Lücke)

1. If $\tau < 2^\kappa \rightarrow \tau^{<\kappa} < 2^\kappa$, then $\text{SMP}_1(\kappa)$ iff $\text{GMA}_\kappa$ and $\kappa^{<\kappa} = \kappa$.

2. $\text{SMP}_2(\kappa)$ implies that $2^\kappa$ is weakly inaccessible, and for all $\tau < 2^\kappa$, $\tau^{<\kappa} < 2^\kappa$.

3. $\text{SMP}_2(\kappa)$ implies that every $\Sigma^1_1$ subset of $\kappa^\kappa$ of cardinality $2^\kappa$ contains a perfect set.

Here:

$A \subset \kappa^\kappa$ contains a perfect set if there is a continuous injection $g : 2^\kappa \rightarrow \kappa^\kappa$ with $\text{ran}(g) \subset A$.

$A \subset \kappa^\kappa$ is $\Sigma^1_1$ iff $A = p[T]$ for some tree $T \subset \kappa^{<\kappa} \times \kappa^{<\kappa}$. 
a proof of

$SMP_2(\kappa)$ implies that every $\Sigma^1_1$ subset of $\kappa^\kappa$ of cardinality $2^\kappa$ contains a perfect set.

proof:
Let $T$ be a tree in $\kappa^{<\kappa} \times \kappa^{<\kappa}$, we look at $p[T]$.
Set $\nu := 2^\kappa$, and let $\dot{Q}$ be an $Add(\kappa, \nu^+)$ name for a $\kappa$-closed, $\kappa^+$ c.c poset. Denote $W := V[Add(\kappa, \nu^+)][\dot{Q}]$.
Note that $V$ and $W$ have the same cardinals.
Two cases:

1. $(p[T])^V \subsetneq (p[T])^W$, or
2. $W \models |p[T]| < 2^\kappa$

Case (1): can construct an embedding $g : 2^{<\kappa} \rightarrow \kappa^{<\kappa} \times \kappa^{<\kappa}$, $\text{ran}(g) \subset T$ that witnesses $p[T]$ contains a perfect set.

So, $\phi := "|p[T]| < 2^\kappa \text{ or there is such an embedding }"$ holds in $W$.
By $SMP_2(\kappa)$, $\phi$ holds in $V$. 
Theorem

From a Mahlo cardinal, it is consistent that $2^{\aleph_0} < \mathcal{B} = 2^{\aleph_1}$, for every $\kappa < 2^{\aleph_1}$, there is a Kurepa tree with at least $\kappa$ many branches, but there is no Kurepa tree with $2^{\aleph_1}$ many branches.

Proof.

Let $V$ be a model of $SMP_2(\omega_1)$ (can be forced from a Mahlo). By the above, in $V$ we have:

- $GMA_{\omega_1}$;
- CH, $2^{\omega_1}$ is weakly inaccessible.
- Every $\Sigma^1_1$ subset of $\omega_1^{\omega_1}$ of cardinality $2^{\omega_1}$ contains a perfect set.
Kurepa trees with (at least) \( \kappa \) many branches for all \( \kappa < 2^{\aleph_1} \):

1. Let \( P \) be the standard poset to add such a tree.
2. \( P \) satisfies the hypothesis of \( GMA_{\omega_1} \);
3. We need only \( \kappa \) many dense sets to meet to get the tree with \( \kappa \) branches.

So by \( GMA_{\omega_1} \), there is a Kurepa tree with at least \( \kappa \) many branches.
No Kurepa trees with $2^\aleph_1$ many branches:

Let $T$ be a Kurepa tree. Look at the set of branches, $[T]$.

Claim: $[T]$ is a closed set that does not contain a perfect set.

Pf:

- Let $g : 2^{\omega_1} \to 2^{\omega_1}$ be a continuous injection with $\text{ran}(g) \subset [T]$.
- Construct $\langle p_s \mid s \in 2^{<\omega} \rangle$ and $\langle \alpha_n \mid n < \omega \rangle$, s.t.
  - $s' \supset s \rightarrow p_{s'} <_T p_s$; $|s| = n \rightarrow \alpha_n = \text{dom}(p_s)$,
  - for each $s$, $p_{s^0} \neq p_{s^1}$.

  by induction on $|s|$.

- Then for $\alpha := \sup_n \alpha_n$, the $\alpha$-th level of $T$ has $2^\omega$ many nodes:
  - for $\eta \in 2^\omega$, set $p_\eta = \bigcup p_{\eta|n}$.

- Contradiction with $T$ being Kurepa.

So $|[T]| < 2^{\omega_1}$, as desired.
1. The idea of using Kurepa trees to get counterexamples to the perfect set property goes back to Mekler and Väänänen.

2. A slightly weaker large cardinals hypothesis than a Mahlo suffices.

3. Our results generalize to $\kappa$-Kurepa trees for $\kappa \geq \aleph_2$. 
Thm: can force $\mathcal{B} = 2^{\aleph_1}$ is not a maximum.

Corollary

*There is an $\mathcal{L}_{\omega_1,\omega}$ sentence $\phi$, such that under some mild large cardinals, it is consistent that the spectrum of $\phi$ is $[\aleph_0, 2^{\aleph_1})$, $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_1}$ is weakly inaccessible.*

Corollary

*Can have $2^{\aleph_0} < \aleph_{\omega_1} < 2^{\aleph_1}$ and sentence with models in $\aleph_{\omega_1}$, but no models in $2^{\aleph_1}$.*

Recall Shelah’s conjecture: If $\aleph_{\omega_1} < 2^{\aleph_0}$, then any $\mathcal{L}_{\omega_1,\omega}$ sentence which has models of size $\aleph_{\omega_1}$ also has models of size $2^{\aleph_0}$.

Corollary

$2^{\aleph_0}$ cannot be replaced by $2^{\aleph_1}$ in the above.
Using consistency results about Kurepa trees, we produce an $\mathcal{L}_{\omega_1,\omega}$ sentence $\phi$, for which it is consistent that:

1. $\phi$ characterizes $2^\aleph_0$
   (take a model with no Kurepa trees or with $\mathcal{B} < 2^\aleph_0$),

2. $\phi$ characterizes $\aleph_\omega$ and $2^\aleph_0 < \aleph_\omega$
   (take the model with $2^\aleph_0 < \mathcal{B} = \aleph_\omega < 2^{\aleph_1}$).

3. $Spec(\phi) = [\aleph_0, 2^{\aleph_1})$ and $2^\aleph_0 < 2^{\aleph_1}$ and the latter is weakly inaccessible
   (use the last theorem, with $\mathcal{B} = 2^{\aleph_1}$ not a maximum).
More on $\phi$

We get similar corollaries regarding the **maximal model** spectrum of $\phi$, $MM - Spec(\phi) := \{ \kappa | \exists M \models \kappa, |M| = \kappa, M \text{ is maximal} \}$, and the **amalgamation spectrum** of $\phi$, $AP - Spec(\phi)$:

It is consistent that:

1. $MM - Spec(\phi) = \{ \aleph_1, 2^{\aleph_0} \}$, $AP - Spec(\phi) = [\aleph_1, 2^{\aleph_0}]$ (take a model with no Kurepa trees),
2. $2^{\aleph_0} < \aleph_{\omega_1}$ and $AP - Spec(\phi) = [\aleph_1, \aleph_{\omega_1}]$;
3. $MM - Spec(\phi)$ is a cofinal subset of $[\aleph_1, 2^{\aleph_1})$, $AP - Spec(\phi) = [\aleph_1, 2^{\aleph_1})$ (use the last theorem, with $B = 2^{\aleph_1}$ not a maximum).
Open questions

1. Shelah’s conjecture: If $\aleph_{\omega_1} < 2^{\aleph_0}$, then any $\mathcal{L}_{\omega_1,\omega}$ sentence which has models of size $\aleph_{\omega_1}$ also has models of size $2^{\aleph_0}$.

2. Recall, model existence in $\aleph_1$ is absolute for $\mathcal{L}_{\omega_1,\omega}$ sentences. **Open**: what about $\aleph_1$-amalgamation for $\mathcal{L}_{\omega_1,\omega}$ sentences? (By Shoenfield $\aleph_0$-amalgamation is absolute)