Preference Logic and Its Measurement-Theoretic Semantics
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Abstract. How can we explain why we should ascribe some logical properties to an agent's preferences, and why we should not ascribe others to them? Although this problem is very important, little attention has been given to it. We break this problem down into the three:

Problem 1. When we attempt to ascribe some logical properties to the preferences of an agent, how can we explain this ascription?

Problem 2. What logical properties must we ascribe to his preferences if and only if our solution to Problem 1 can be adopted?

Problem 3. What logical properties should we ascribe to his preferences and what logical properties should not we ascribe to them?

The aim of this paper is to propose a new version of sound and complete preference logic (PL) that can furnish a solution to each of these problems. We solve all of them by providing PL with a Domotor-type semantics that is a kind of measurement-theoretic and decision-theoretic one.

Key Words: preference logic, measurement theory, representation theorem, decision theory, projective geometry, separation theorem.

1 Introduction

The notion of preference plays an important role in many disciplines, including philosophy and economics.[1] Some of notable recent developments in ethics make substantial use of preference logic.[2] In computer science, preference logic has become an indispensable device. The founder of preference logic is the founding father of logic itself, Aristotle. Book III of the *Topics* can be regarded as the first treatment of the subject. From the 1950s to the 1960s, the study of

[1] [8] gives a comprehensive survey of preference in general.
[2] [7] gives a comprehensive survey of preference logic.

preference logic flourished in Scandinavia–particularly by Halldén ([6]) and von Wright ([26]) and in the U.S.A.–particularly by Martin ([17]) and Chisholm and Sosa ([4]). Recently with the help of Boutilier's idea ([3]), van Benthem, Otterloo and Roy reduced preference logic to multi-modal logic ([25]). Some logical properties are provable in one preference logic, but they are not provable in another preference logic. For example, the status of such logical properties as (transitivity), (contraposition), (conjunctive expansion), (disjunctive distribution) and (conjunctive distribution) is as follows:

Example 1.

	von Wright	Martin	Chisholm and Sosa
Transitivity	+	+	+
Contraposition	−	+	−
Conjunctive Expansion	+	−	−
Disjunctive Distribution	−	−	−
Conjunctive Distribution	+	−	−

'+' denotes the property in question being provable in the logic in question. '−' denotes the property in question not being provable in the logic in question. (Conjunctive expansion) says that an agent does not prefer φ_1 to φ_2 iff he does not prefer $\varphi_1 \wedge \neg\varphi_2$ to $\varphi_2 \wedge \neg\varphi_1$. (Disjunctive distribution) says that if he does not prefer $\varphi_1 \vee \varphi_2$ to φ_3, then he does not prefer φ_1 to φ_3 or does not prefer φ_2 to φ_3. (Conjunctive distribution) says that if he does not prefer φ_1 to φ_2 and does not prefer φ_3 to φ_2, then he does not prefer $\varphi_1 \vee \varphi_3$ to φ_2.

Then how we can explain why we should ascribe some logical properties to an agent's preferences, and why we should not ascribe others to them? Although this problem is very important, little attention has been given to it. We break this problem down into the three:

Problem 1. **When we attempt to ascribe some logical properties to the preferences of an agent, how can we explain this ascription?**

Problem 2. **What logical properties must we ascribe to his preferences if and only if our solution to Problem 1 can be adopted?**

Problem 3. **What logical properties should we ascribe to his preferences and what logical properties should not we ascribe to them?**

The aim of this paper is to propose a new version of sound and complete preference logic (PL) that can furnish a solution to each of these problems. We solve all of them by providing PL with a Domotor-type semantics that is a kind of measurement-theoretic and decision-theoretic one.

Measurement theory is one that provides measurement with its mathematical foundation.[3] The mathematical foundation of measurement had not been studied before Hölder developed his axiomatisation for the measurement of mass ([9]). [14], [24] and [16] are seen as milestones in the history of measurement theory. In measurement theory, at least four kinds of measurement have been objects of study:

[3] [21] gives a comprehensive survey of measurement theory.

1. ordinal measurement,
2. extensive measurement,
3. difference measurement,
4. conjoint measurement.

On the other hand, there are at least two kinds of decision theory:

1. evidential decision theory,[4]
2. causal decision theory.[5]

The former is designed for decision makings that have statistical or evidential connections between actions and outcomes. The latter is designed for decision makings that have causal connections between actions and outcomes. Both theories take the form of subjective expected utility theory. Jeffrey ([11]) is a typical example of the former. Ramsey ([19]) is a typical example of the latter. Ramsey regarded desire as attitude toward outcomes but belief as one toward propositions. Moreover, he regarded preference as attitude toward an ordered pair of gambles, that is, hybrid entities composed of outcomes and propositions. In 1965 Jeffrey ([11]) developed an alternative to Ramsey's theory. He regarded both desire and belief as attitudes toward propositions. Moreover, he regarded preference as attitude toward an ordered pair of propositions. In this sense we call Jeffrey's a mono-set theory. Its initial axiomatisation was provided in terms of measurement theory by Bolker ([2]) on the mathematics developed in [1]. Jeffrey ([10]) modified Bolker's axioms to accommodate null propositions. Domotor ([5]) also axiomatised a version of mono-set theory. Mono-set theories are more suitable for the semantics of logic than Ramsey's, for regarding propositions as the semantic values of sentences is simpler than regarding gambles as those when we wish to provide logic with its semantics. Especially, Domotor's theory is the most suitable for the semantics of logic of these three mono-set theories, for constructing the syntactic analogues of the axioms of Domotor's theory is easier than of the other two theories.

Like Bolker's and Jeffrey's, Domotor's theory has a conjoint structure. In them, preferences are decomposable into beliefs and desires. From a measurement-theoretic viewpoint of decision theory, there is a tradition to specify or explain an agent's beliefs and desires in terms of his preferences [and vice versa]. This specification takes the form of a representation theorem:

> If [and only if] an agent's preferences satisfy such-and-such conditions, there exist a probability function and a utility function such that he should act as an expected utility maximiser (existence). [In addition, the pair of such probability function and utility function is unique up to a kind of transformation (uniqueness).]

Domotor's representation theorem is the only known one that can furnish conditions of an agent's preferences necessary and sufficient for there existing a

[4] [11] gives a comprehensive survey of evidential decision theory.
[5] [13] gives a comprehensive survey of causal decision theory.

probability function and a utility function such that he should act as an expected utility maximiser. All other representation theorems, such as [2] and [10], can furnish only sufficient conditions for it. So if an agent's belief state can be represented by a probability function and his desire state can be represented by a utility function, then only by virtue of Domotor's representation theorem, we can explain ascribing some logical properties to his preferences in terms of his beliefs and desires via expected utility maximisation, which can furnish a solution to Problem 1. But by virtue of all other representation theorems, we cannot do so. By virtue of Domotor's representation theorem, we should ascribe (connectedness) and (projectivity) to an agent's preferences if and only if our solution to Problem 1 can be adopted, which can furnish a solution to Problem 2. Generally, conjoint measurement requires the cancellation axiom as a necessary one. (Projectivity) can be regarded as a generalisation of the cancellation axiom. Domotor's representation theorem follows from Scott's separation theorem ([22]). The latter is based on the general mathematical criterion for the solvability of a finite set of homogeneous linear inequalities.

The structure of this paper is as follows. In Section 2, we prepare the projective-geometric concepts for the measurement-theoretic settings: characteristic function, exterior product, symmetric product and four-fold exterior product, and define preference space and preference assignment, and state necessary and sufficient conditions for representation: (connectedness) and (projectivity), and prove a representation theorem. In Section 3, we define the language $\mathcal{L}_{\mathsf{PL}}$ of PL, and define a Domotor-type structured Kripke model $\mathcal{M}$ for preference, and provide PL with a truth definition, and provide PL with a proof system, and prove the soundness of PL in the usual way, and prove the completeness of PL by constructing the canonical model, and prove that (reflexivity), (transitivity), (connectedness) and (impartiality) are all provable in PL, and that neither (contraposition), (conjunctive expansion), (disjunctive distribution) nor (conjunctive distribution) is provable in PL, but that (restricted contraposition), (restricted conjunctive expansion), (restricted disjunctive distribution) and (restricted conjunctive distribution) are all provable in PL, which can furnish a solution to Problem 3.

2 Measurement-Theoretic Settings

2.1 Projective-Geometric Concepts

We define the preliminaries to the measurement-theoretic setting.

Definition 1 (Preliminaries). $\mathbf{W}$ *is a nonempty set of possible worlds. Let* $\mathcal{F}$ *denote a Boolean field of subsets of* $\mathbf{W}$*. We call* $A \in \mathcal{F}$ *a proposition.*

Because it is impossible to characterise multiplication of probabilities and utilities in terms of union, intersection and preferences, we need a Cartesian product $\times$. A characteristic function is definable also on a Cartesian product of propositions. We define a characteristic function as follows:

Definition 2 (Characteristic Function). *A characteristic function* $\widehat{\ }: \mathcal{F} \to \{0,1\}^{\mathbf{W}}$ *is one where for any* $A \in \mathcal{F}$ *we have* $\widehat{A}: \mathbf{W} \to \{0,1\}$ *such that*

$$\widehat{A}(w) := \begin{cases} 1 & \text{if } w \in A, \\ 0 & \text{otherwise,} \end{cases}$$

for any $w \in \mathbf{W}$. *A Cartesian product of characteristic functions* $\otimes$ *is defined as follows:* $\widehat{A} \otimes \widehat{B} := \widehat{(A \times B)}$.

By means of $\otimes$ we define an exterior product $\circ$ as follows:

Definition 3 (Exterior Product).
$\widehat{A} \circ \widehat{B} := \widehat{A} \otimes \widehat{B} - \widehat{B} \otimes \widehat{A} = \widehat{(A \times B)} - \widehat{(B \times A)}$.

By means of $\circ$ we define a symmetric product $\odot$ as follows:

Definition 4 (Symmetric Product).

$$\begin{aligned}
&\odot(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}) \\
&:= (\widehat{A} \circ \widehat{B}) \circ (\widehat{C} \circ \widehat{D}) + (\widehat{C} \circ \widehat{D}) \circ (\widehat{A} \circ \widehat{B}) = \\
&\widehat{(A \times B \times C \times D)} + \widehat{(B \times A \times D \times C)} + \widehat{(C \times D \times A \times B)} + \widehat{(D \times C \times B \times A)} \\
&- \widehat{(A \times B \times D \times C)} - \widehat{(B \times A \times C \times D)} - \widehat{(C \times D \times B \times A)} - \widehat{(D \times C \times A \times B)}.
\end{aligned}$$

By means of $\odot$ we define a four-fold exterior product $\triangle$ as follows:

Definition 5 (Four-Fold Exterior Product).

$$\begin{aligned}
&\triangle(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}) := \\
&\odot(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}) + \odot(\widehat{A}, \widehat{C}, \widehat{D}, \widehat{B}) + \odot(\widehat{A}, \widehat{D}, \widehat{B}, \widehat{C}) = \\
&\widehat{(A \times B \times C \times D)} + \widehat{(B \times A \times D \times C)} + \widehat{(C \times D \times A \times B)} + \widehat{(D \times C \times B \times A)} \\
&- \widehat{(A \times B \times D \times C)} - \widehat{(B \times A \times C \times D)} - \widehat{(C \times D \times B \times A)} - \widehat{(D \times C \times A \times B)} \\
&+ \widehat{(A \times C \times D \times B)} + \widehat{(C \times A \times B \times D)} + \widehat{(D \times B \times A \times C)} + \widehat{(B \times D \times C \times A)} \\
&- \widehat{(A \times C \times B \times D)} - \widehat{(C \times A \times D \times B)} - \widehat{(D \times B \times C \times A)} - \widehat{(B \times D \times A \times C)} \\
&+ \widehat{(A \times D \times B \times C)} + \widehat{(D \times A \times C \times B)} + \widehat{(B \times C \times A \times D)} + \widehat{(C \times B \times D \times A)} \\
&- \widehat{(A \times D \times C \times B)} - \widehat{(D \times A \times B \times C)} - \widehat{(B \times C \times D \times A)} - \widehat{(C \times B \times A \times D)}
\end{aligned}$$

2.2 Preference Space and Preference Assignment

We define preference space and preference assignment as follows:

Definition 6 (Preference Space and Preference Assignment). $\preceq_w$ *is a weak preference relation on* $\mathcal{F}^2$. $A \preceq_w B$ *is interpreted to mean that the agent does not prefer* A *to* B *at a time in* w. $\sim_w$ *and* $\prec_w$ *are defined as follows:*

- $A \sim_w B := A \preceq_w B$ *and* $B \preceq_w A$,
- $A \prec_w B := A \preceq_w B$ *and* $A \not\sim_w B$.

For any $w \in \mathbf{W}$, $(\mathbf{W}, \mathcal{F}, \preceq_w)$ *is called a preference space. Let* $\mathbf{PS}$ *denote the set of all preference spaces.* $\rho : \mathbf{W} \to \mathbf{PS}$ *is called a preference assignment.*

2.3 Conditions for Representation

We can state necessary and sufficient conditions for representation as follows:

1. $A \preceq_w B$ or $B \preceq_w A$ (**Connectedness**),
2. If ($A_i \preceq_w B_i$ and $C_i \preceq_w D_i$ for any $i < n$),
   then (if $A_n \preceq_w B_n$, then $D_n \preceq_w C_n$),
   where $\sum_{i \le n} \odot(\widehat{A_i}, \widehat{B_i}, \widehat{C_i}, \widehat{D_i}) = \triangle(\widehat{A_n}, \widehat{B_n}, \widehat{C_n}, \widehat{D_n})$ (**Projectivity**).

2.4 Explanation for Projectivity

Under Domotor's representation theorem, (projectivity) essentially says that if

$$\sum_{i=1}^{n} P_w(A_i)P_w(B_i)P_w(C_i)P_w(D_i)(U_w(B_i) - U_w(A_i))(U_w(D_i) - U_w(C_i)) = 0$$

and if $U_w(A_i) \le U_w(B_i)$ for $i = 1, \ldots, n$ and $U_w(C_i) \le U_w(D_i)$ for $i = 1, \ldots, n-1$, then $U_w(D_n) \le U_w(C_n)$. Zero on the right hand side comes from the fact that the measure of $\triangle(\widehat{A_n}, \widehat{B_n}, \widehat{C_n}, \widehat{D_n})$ happens to be equal to zero:

$$P_w(A_n)P_w(B_n)P_w(C_n)P_w(D_n)((U_w(B_n) - U_w(A_n))(U_w(D_n) - U_w(C_n))$$
$$+(U_w(C_n) - U_w(A_n))(U_w(B_n) - U_w(D_n)) + (U_w(D_n) - U_w(A_n))(U_w(C_n) - U_w(B_n))) = 0.$$

2.5 Domotor's Representation Theorem

We can prove Domotor's representation theorem as follows:

Theorem 1 (Representation). *For any $w \in \mathbf{W}$, $(\mathbf{W}, \mathcal{F}, \preceq_w, \widehat{\ }, \times)$ satisfies (connectedness) and (projectivity) iff there are $P_w : \mathcal{F} \to \mathbb{R}$ and $U_w : \mathcal{F}\backslash\emptyset \to \mathbb{R}$ such that the following conditions hold for any $A, B \in \mathcal{F}\backslash\emptyset$:*

- $(\mathbf{W}, \mathcal{F}, P_w)$ *is a finitely additive probability space,*
- $A \preceq_w B$ *iff* $U_w(A) \le U_w(B)$,
- $U_{w_1}(A) = \sum_{w_2 \in A} P_{w_1}(\{w_2\})U_{w_1}(\{w_2\})$,
- *When $A \in \mathcal{F}$, if $P_w(A) = 0$, then $A = \emptyset$.*

Proof. Except that the proof is relative to world, it is similar to that of [[5]:184–194].

2.6 Significance of Domotor's Representation Theorem

Theorem 1 (Domotor's representation theorem) is the only known one that can furnish conditions of an agent's preferences necessary and sufficient for there existing a probability function and a utility function such that he should act as an expected utility maximiser. All other representation theorems, such as [2] and

[10], can furnish only sufficient conditions for it. So if an agent's belief state can be represented by a probability function and his desire state can be represented by a utility function, then only by virtue of Theorem 1, we can explain ascribing some logical properties to his preferences in terms of his beliefs and desires via expected utility maximisation, which can furnish a solution to Problem 1. But by virtue of all other representation theorems, we cannot do so. What has to be noticed is that we never insist that expected utility maximisation should be the best way to make rational decisions. By virtue of Theorem 1, we should ascribe (connectedness) and (projectivity) to an agent's preferences if and only if our solution to Problem 1 can be adopted, which can furnish a solution to Problem 2. In Theorem 1, we do not obtain the uniqueness result. But it does not matter when we provide PL with its semantics.

2.7 Scott's Separation Theorem

Generally, conjoint measurement requires the cancellation axiom as a necessary one. (Projectivity) can be regarded as a generalisation of the cancellation axiom. Domotor's representation theorem follows from Scott's separation theorem.

Theorem 2 (Separation, Scott [22]). *Let I be a finite-dimensional real linear vector space and let $\emptyset \neq G \subset H \subset I$, where $H = -H = \{-v : v \in H\}$ is finite and all its elements have rational coordinates with respect to a given basis. Then there exists a linear functional $F : I \to \mathbb{R}$ such that for any $v \in H$*

$$F(v) \geq 0 \textit{ iff } v \in G$$

iff for any $v, v_i \in H$ $(1 \leq i \leq n)$ we have both

$$(1) \quad v \in G \textit{ or } -v \in G$$

and

$$(2) \quad \textit{If } v_i \in G \textit{ for any } i < n, \textit{ then } -v \in G, \textit{ where } \sum_{i \leq n} v_i = 0.$$

(1) corresponds to (connectedness) and (2) corresponds to (projectivity). Scott's separation theorem is based on the general mathematical criterion for the solvability of a finite set of homogeneous linear inequalities.

3 Preference Logic PL

3.1 Language

The language $\mathcal{L}_{\mathsf{PL}}$ of PL is defined as follows:

Definition 7 (Language). *Let* **S** *denote a set of sentential variables,* **WPR** *a weak preference relation symbol, and* **FCP** *a four-fold Cartesian product symbol. $\mathcal{L}_{\mathsf{PL}}$ is given by the following rule:*

$$\varphi ::= s \mid \top \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \mathbf{WPR}(\varphi_1, \varphi_2) \mid \mathbf{FCP}(\varphi_1, \varphi_2, \varphi_3, \varphi_4),$$

where $s \in \mathbf{S}$. $\bot, \vee, \rightarrow$ and $\leftrightarrow$ are introduced by the standard definitions. **IND** *and* **SPR** *are defined as follows:*

- $\mathbf{IND}(\varphi_1, \varphi_2) := \mathbf{WPR}(\varphi_1, \varphi_2) \wedge \mathbf{WPR}(\varphi_2, \varphi_1)$,
- $\mathbf{SPR}(\varphi_1, \varphi_2) := \mathbf{WPR}(\varphi_1, \varphi_2) \wedge \neg\mathbf{IND}(\varphi_1, \varphi_2)$.

The set of all well-formed formulae of $\mathcal{L}_{\mathsf{PL}}$ will be denoted by $\Phi_{\mathcal{L}_{\mathsf{PL}}}$.

3.2 Semantics

Model By developing the idea of Naumov ([18]), we define a Domotor-type structured Kripke model $\mathcal{M}$ for preference as follows:

Definition 8 (Model). *$\mathcal{M}$ is a quintuple $(\mathbf{W}, R, L, V, \rho)$, where $\mathbf{W}$ is a nonempty set of possible worlds, R is a relation on $\mathbf{W}^2$, $(\boldsymbol{W}, R)$ is a directed acyclic graph, $L : R \rightarrow \{\pi_1, \pi_2, \pi_3, \pi_4\}$ is a function that assigns labels to the edges of the graph, any two edges leaving the same vertex have different labels, any vertex either has π_1-, π_2-, π_3- and π_4-labeled outgoing edges or none of them, V is a truth assignment to each $s \in S$ for each $w \in \mathbf{W}$, and ρ is a preference assignment that assigns to each $w \in \mathbf{W}$ $(\mathbf{W}, \mathcal{F}, \preceq_w)$ that satisfies (connectedness) and (projectivity). For any $w_1 \in \mathbf{W}$, by $\pi_i(w_1)$ $(i = 1, 2, 3, 4)$ we mean the unique $w_2 \in \mathbf{W}$ such that $R(w_1, w_2)$ and $L(w_1, w_2) = \pi_i$ if such world exists.*

Truth Definition We can provide PL with the following truth definition:

Definition 9 (Truth). *The notion of $\varphi \in \Phi_{\mathcal{L}_{\mathsf{PL}}}$ being true at $w \in W$ in $\mathcal{M}$, in symbols $(\mathcal{M}, w) \models_{\mathsf{PL}} \varphi$ is inductively defined as follows:*

- $(\mathcal{M}, w) \models_{\mathsf{PL}} s$ *iff* $V(w)(s) = \mathbf{true}$,
- $(\mathcal{M}, w) \models_{\mathsf{PL}} \top$ *iff for all w,* $V(w)(\top) = \mathbf{true}$,
- $(\mathcal{M}, w) \models_{\mathsf{PL}} \varphi_1 \wedge \varphi_2$ *iff* $(\mathcal{M}, w) \models_{\mathsf{PL}} \varphi_1$ *and* $(\mathcal{M}, w) \models_{\mathsf{PL}} \varphi_2$,
- $(\mathcal{M}, w) \models_{\mathsf{PL}} \neg\varphi$ *iff* $(\mathcal{M}, w) \not\models_{\mathsf{PL}} \varphi$,
- $(\mathcal{M}, w) \models_{\mathsf{PL}} \mathbf{FCP}(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ *iff* $(\mathcal{M}, \pi_1(w)) \models_{\mathsf{PL}} \varphi_1$ *and* $(\mathcal{M}, \pi_2(w)) \models_{\mathsf{PL}} \varphi_2$ *and* $(\mathcal{M}, \pi_3(w)) \models_{\mathsf{PL}} \varphi_3$ *and* $(\mathcal{M}, \pi_4(w)) \models_{\mathsf{PL}} \varphi_4$,
- $(\mathcal{M}, w) \models_{\mathsf{PL}} \mathbf{WPR}(\varphi_1, \varphi_2)$ *iff* $[\![\varphi_1]\!] \preceq_w [\![\varphi_2]\!]$,

where $[\![\varphi]\!] := \{w \in \mathbf{W} : (\mathcal{M}, w) \models_{\mathsf{PL}} \varphi\}$. If $(\mathcal{M}, w) \models_{\mathsf{PL}} \varphi$ for all $w \in \mathbf{W}$, we write $\mathcal{M} \models_{\mathsf{PL}} \varphi$ and say that φ is valid in $\mathcal{M}$. If φ is valid in all Domotor-type structured model for preference, we write $\models_{\mathsf{PL}} \varphi$ and say that φ is valid.

3.3 Syntax

Preliminaries We devise a syntactic analogue of (projectivity). By developing the idea of Segerberg ([23]), we define Γ_i as follows:

Definition 10 (Disjunction of Conjunctions). *For any i $(1 \leq i \leq 4n+4)$, Γ_i is defined as the disjunction of all the following conjunctions:*

$$
\begin{aligned}
&\bigwedge_{j=1}^{n-1} d_j\mathbf{FCP}(\varphi_j, \psi_j, \chi_j, \tau_j)\\
&\wedge d_n\mathbf{FCP}(\varphi_n, \chi_n, \psi_n, \tau_n)\\
&\wedge d_{n+1}\mathbf{FCP}(\varphi_n, \tau_n, \chi_n, \psi_n)\\
&\wedge \bigwedge_{j=n+2}^{2n} d_j\mathbf{FCP}(\psi_{j-n-1}, \varphi_{j-n-1}, \tau_{j-n-1}, \chi_{j-n-1})\\
&\wedge d_{2n+1}\mathbf{FCP}(\chi_n, \varphi_n, \tau_n, \psi_n)\\
&\wedge d_{2n+2}\mathbf{FCP}(\tau_n, \varphi_n, \psi_n, \chi_n)\\
&\wedge \bigwedge_{j=2n+3}^{3n+1} d_j\mathbf{FCP}(\chi_{j-2n-2}, \tau_{j-2n-2}, \varphi_{j-2n-2}, \psi_{j-2n-2})\\
&\wedge d_{3n+2}\mathbf{FCP}(\tau_n, \psi_n, \chi_n, \varphi_n)\\
&\wedge d_{3n+3}\mathbf{FCP}(\psi_n, \chi_n, \tau_n, \varphi_n)\\
&\wedge \bigwedge_{j=3n+4}^{4n+2} d_j\mathbf{FCP}(\tau_{j-3n-3}, \chi_{j-3n-3}, \psi_{j-3n-3}, \varphi_{j-3n-3})\\
&\wedge d_{4n+3}\mathbf{FCP}(\psi_n, \tau_n, \varphi_n, \chi_n)\\
&\wedge d_{4n+4}\mathbf{FCP}(\chi_n, \psi_n, \varphi_n, \tau_n)\\
&\wedge \bigwedge_{j=1}^{n-1} e_j\mathbf{FCP}(\varphi_j, \psi_j, \tau_j, \chi_j)\\
&\wedge e_n\mathbf{FCP}(\varphi_n, \chi_n, \tau_n, \psi_n)\\
&\wedge e_{n+1}\mathbf{FCP}(\varphi_n, \tau_n, \psi_n, \chi_n)\\
&\wedge \bigwedge_{j=n+2}^{2n} e_j\mathbf{FCP}(\psi_{j-n-1}, \varphi_{j-n-1}, \chi_{j-n-1}, \tau_{j-n-1})\\
&\wedge e_{2n+1}\mathbf{FCP}(\chi_n, \varphi_n, \psi_n, \tau_n)\\
&\wedge e_{2n+2}\mathbf{FCP}(\tau_n, \varphi_n, \chi_n, \psi_n)\\
&\wedge \bigwedge_{j=2n+3}^{3n+1} e_j\mathbf{FCP}(\chi_{j-2n-2}, \tau_{j-2n-2}, \psi_{j-2n-2}, \varphi_{j-2n-2})\\
&\wedge e_{3n+2}\mathbf{FCP}(\tau_n, \psi_n, \varphi_n, \chi_n)\\
&\wedge e_{3n+3}\mathbf{FCP}(\psi_n, \chi_n, \varphi_n, \tau_n)\\
&\wedge \bigwedge_{j=3n+4}^{4n+2} e_j\mathbf{FCP}(\tau_{j-3n-3}, \chi_{j-3n-3}, \varphi_{j-3n-3}, \psi_{j-3n-3})\\
&\wedge e_{4n+3}\mathbf{FCP}(\psi_n, \tau_n, \chi_n, \varphi_n)\\
&\wedge e_{4n+4}\mathbf{FCP}(\chi_n, \psi_n, \tau_n, \varphi_n)
\end{aligned}
$$

such that exactly i of the d_j's and i of the e_j's are the negation symbols, the rest of them being the empty string of symbols.

By means of Γ_i, we define **DDC** as follows:

Definition 11 (Disjunction of Disjunctions of Conjunctions).

$$\mathbf{DDC}_{i=1}^{n}(\varphi_i, \psi_i, \chi_i, \tau_i) := \vee_{i=1}^{4n+4} \Gamma_i$$

Proof System We provide PL with the following proof system.

Definition 12 (Proof System).

- ***Axioms of*** PL

(*A1*) *All tautologies of classical sentential logic,*

(*A2*) $\mathbf{WPR}(\varphi_1, \varphi_2) \vee \mathbf{WPR}(\varphi_2, \varphi_1)$ (***Syntactic Analogue of Connectedness***),

(*A3*) $\mathbf{DDC}_{i=1}^{n}(\varphi_i, \psi_i, \chi_i, \tau_i) \to$
$(\wedge_{i=1}^{n}(\mathbf{WPR}(\varphi_i, \psi_i) \wedge \mathbf{WPR}(\chi_i, \tau_i)) \to (\mathbf{WPR}(\varphi_n, \psi_n) \to \mathbf{WPR}(\tau_n, \chi_n)))$
(***Syntactic Analogue of Projectivity***),

(*A4*) $\mathbf{FCP}(\top, \top, \top, \top)$ (***Tautology and Four-Fold Cartesian Product***),

(*A5*) $\mathbf{FCP}(\varphi_1 \wedge \varphi_2, \psi_1 \wedge \psi_2, \chi_1 \wedge \chi_2, \tau_1 \wedge \tau_2) \to (\mathbf{FCP}(\varphi_1, \psi_1, \chi_1, \tau_1) \wedge \mathbf{FCP}(\varphi_2, \psi_2, \chi_2, \tau_2))$
(***Conjunction and Four-Fold Cartesian Product 1***),

(*A6*) $(\mathbf{FCP}(\varphi_1, \mu, \nu, \xi) \wedge \mathbf{FCP}(\varphi_2, \mu, \nu, \xi)) \to \mathbf{FCP}(\varphi_1 \wedge \varphi_2, \mu, \nu, \xi)$
(***Conjunction and Four-Fold Cartesian Product 2***),

(*A7*) $(\mathbf{FCP}(\lambda, \psi_1, \nu, \xi) \wedge \mathbf{FCP}(\lambda, \psi_2, \nu, \xi)) \to \mathbf{FCP}(\lambda, \psi_1 \wedge \psi_2, \nu, \xi)$
(***Conjunction and Four-Fold Cartesian Product 3***),

(*A8*) $(\mathbf{FCP}(\lambda, \mu, \chi_1, \xi) \wedge \mathbf{FCP}(\lambda, \mu, \chi_2, \xi)) \to \mathbf{FCP}(\lambda, \mu, \chi_1 \wedge \chi_2, \xi)$
(***Conjunction and Four-Fold Cartesian Product 4***),

(*A9*) $(\mathbf{FCP}(\lambda, \mu, \nu, \tau_1) \wedge \mathbf{FCP}(\lambda, \mu, \nu, \tau_2)) \to \mathbf{FCP}(\lambda, \mu, \nu, \tau_1 \wedge \tau_2)$
(***Conjunction and Four-Fold Cartesian Product 5***),

(*A10*) $\neg\mathbf{FCP}(\varphi, \psi, \chi, \tau)$
$\leftrightarrow (\mathbf{FCP}(\neg\varphi, \psi, \chi, \tau) \vee \mathbf{FCP}(\varphi, \neg\psi, \chi, \tau) \vee \mathbf{FCP}(\varphi, \psi, \neg\chi, \tau) \vee \mathbf{FCP}(\varphi, \psi, \chi, \neg\tau))$
(***Negation and Four-Fold Cartesian Product***).

- ***Inference Rules of*** PL

(*R1*) $\dfrac{\varphi_1 \quad \varphi_1 \to \varphi_2}{\varphi_2}$ (***Modus Ponens***),

(*R2*) $\dfrac{\varphi_1 \to \varphi_2}{\mathbf{WPR}(\varphi_2, \varphi_1)}$ (***Weak Preference Necessitation***),

(*R3*) $\dfrac{\varphi \wedge \psi \wedge \chi \wedge \tau}{\mathbf{FCP}(\varphi, \psi, \chi, \tau)}$ (***Four-Fold Cartesian Product Necessitation***).

A proof of $\varphi \in \Phi_{\mathsf{PL}}$ is a finite sequence of $\mathcal{L}_{\mathsf{PL}}$-formulae having φ as the last formula such that either each formula is an instance of an axiom, or it can be obtained from formulae that appear earlier in the sequence by applying an inference rule. If there is a proof of φ, we write $\vdash_{\mathsf{PL}} \varphi$.

3.4 Soundness and Completeness

We can prove the soundness of PL in the usual way.

Theorem 3 (Soundness). *For every $\varphi \in \Phi_{\mathsf{PL}}$, if $\vdash_{\mathsf{PL}} \varphi$, then $\models_{\mathsf{PL}} \varphi$.*

We can prove the completeness of PL by constructing the canonical model.

Theorem 4 (Completeness). *For every $\varphi \in \Phi_{\mathsf{PL}}$, if $\models_{\mathsf{PL}} \varphi$, then $\vdash_{\mathsf{PL}} \varphi$.*

3.5 Logical Properties of Preference

(Reflexivity), (transitivity), (connectedness) and (impartiality) are all provable in PL.

Proposition 1 (Reflexivity, Transitivity, Connectedness and Impartiality).

- $\vdash_{\mathsf{PL}} \mathbf{WPR}(\varphi, \varphi)$ (***Reflexivity***),

- $\vdash_{\mathsf{PL}} \mathbf{WPR}((\varphi_1, \varphi_2) \wedge \mathbf{WPR}(\varphi_2, \varphi_3)) \rightarrow \mathbf{WPR}(\varphi_1, \varphi_3)$ (***Transitivity***),

- $\vdash_{\mathsf{PL}} \mathbf{WPR}(\varphi_1, \varphi_2) \vee \mathbf{WPR}(\varphi_2, \varphi_1)$ (***Connectedness***),

- $\vdash_{\mathsf{PL}} ((\mathbf{SPR}(\varphi_2, \varphi_3) \wedge \mathbf{SPR}(\varphi_2, \varphi_4)) \vee (\mathbf{SPR}(\varphi_3, \varphi_1) \wedge \mathbf{SPR}(\varphi_4, \varphi_1)))$
$\rightarrow ((\mathbf{IND}(\varphi_1, \varphi_2) \wedge ((\varphi_1 \wedge \varphi_3) \leftrightarrow (\varphi_2 \wedge \varphi_3) \leftrightarrow (\varphi_1 \wedge \varphi_4) \leftrightarrow (\varphi_2 \wedge \varphi_4) \leftrightarrow \bot))$
$\rightarrow (\mathbf{WPR}(\varphi_1 \vee \varphi_3, \varphi_2 \vee \varphi_3) \leftrightarrow \mathbf{WPR}(\varphi_1 \vee \varphi_4, \varphi_2 \vee \varphi_4)))$ (***Impartiality 1***),

- $\vdash_{\mathsf{PL}} ((\mathbf{SPR}(\varphi_3, \varphi_2) \wedge \mathbf{SPR}(\varphi_2, \varphi_4)) \vee (\mathbf{SPR}(\varphi_4, \varphi_2) \wedge \mathbf{SPR}(\varphi_2, \varphi_3)))$
$\rightarrow ((\mathbf{IND}(\varphi_1, \varphi_2) \wedge ((\varphi_1 \wedge \varphi_3) \leftrightarrow (\varphi_2 \wedge \varphi_3) \leftrightarrow (\varphi_2 \wedge \varphi_4) \leftrightarrow (\varphi_1 \wedge \varphi_4) \leftrightarrow \bot))$
$\rightarrow (\mathbf{WPR}(\varphi_1 \vee \varphi_3, \varphi_2 \vee \varphi_3) \leftrightarrow \mathbf{WPR}(\varphi_2 \vee \varphi_4, \varphi_1 \vee \varphi_4)))$ (***Impartiality 2***),

- $\vdash_{\mathsf{PL}} (\neg\mathbf{IND}(\varphi_2, \varphi_3) \wedge \neg\mathbf{IND}(\varphi_2, \varphi_4))$
$\rightarrow ((\mathbf{IND}(\varphi_1, \varphi_2) \wedge ((\varphi_1 \wedge \varphi_3) \leftrightarrow (\varphi_2 \wedge \varphi_3) \leftrightarrow (\varphi_1 \wedge \varphi_4) \leftrightarrow (\varphi_2 \wedge \varphi_4) \leftrightarrow \bot))$
$\rightarrow (\mathbf{IND}(\varphi_1 \vee \varphi_3, \varphi_2 \vee \varphi_3) \leftrightarrow \mathbf{IND}(\varphi_1 \vee \varphi_4, \varphi_2 \vee \varphi_4)))$ (***Impartiality 3***).

Neither (contraposition), (conjunctive expansion), (disjunctive distribution) nor (conjunctive distribution) is provable in PL.

Proposition 2 (Contraposition, Conjunctive Expansion, Disjunctive Distribution and Conjunctive Distribution).

- $\nvdash_{\mathsf{PL}} \mathbf{WPR}(\varphi_1, \varphi_2) \leftrightarrow \mathbf{WPR}(\neg\varphi_2, \neg\varphi_1)$ (***Contraposition***),

- $\nvdash_{\mathsf{PL}} \mathbf{WPR}(\varphi_1, \varphi_2) \leftrightarrow \mathbf{WPR}(\varphi_1 \wedge \neg\varphi_2, \varphi_2 \wedge \neg\varphi_1)$ (***Conjunctive Expansion***),

- $\nvdash_{\mathsf{PL}} \mathbf{WPR}(\varphi_1 \vee \varphi_2, \varphi_3) \rightarrow (\mathbf{WPR}(\varphi_1, \varphi_3) \vee \mathbf{WPR}(\varphi_2, \varphi_3))$
(***Disjunctive Distribution of Left Disjunction***),

- $\nvdash_{\mathsf{PL}} \mathbf{WPR}(\varphi_1, \varphi_2 \vee \varphi_3) \rightarrow (\mathbf{WPR}(\varphi_1, \varphi_2) \vee \mathbf{WPR}(\varphi_1, \varphi_3))$
(***Disjunctive Distribution of Right Disjunction***),

- $\nvdash_{\mathsf{PL}} (\mathbf{WPR}(\varphi_1, \varphi_2) \wedge \mathbf{WPR}(\varphi_3, \varphi_2)) \rightarrow \mathbf{WPR}(\varphi_1 \vee \varphi_3, \varphi_2)$
(***Conjunctive Distribution of Left Disjunction***),

- $\nvdash_{\mathsf{PL}} (\mathbf{WPR}(\varphi_1, \varphi_2) \wedge \mathbf{WPR}(\varphi_1, \varphi_3)) \rightarrow \mathbf{WPR}(\varphi_1, \varphi_2 \vee \varphi_3)$
(***Conjunctive Distribution of Right Disjunction***).

(Restricted contraposition), (restricted conjunctive expansion), (restricted disjunctive distribution) and (restricted conjunctive distribution) are all provable in PL.

Proposition 3 (Restricted Contraposition, Restricted Conjunctive Expansion, Restricted Disjunctive Distribution and Restricted Conjunctive Distribution).

- $\vdash_{\mathsf{PL}} ((\varphi_1 \wedge \varphi_2) \leftrightarrow \bot) \rightarrow (\mathbf{WPR}(\varphi_1, \varphi_2) \leftrightarrow \mathbf{WPR}(\neg\varphi_2, \neg\varphi_1))$
(***Restricted Contraposition***),

- $\vdash_{\mathsf{PL}} ((\varphi_1 \wedge \varphi_2) \leftrightarrow \bot) \rightarrow (\mathbf{WPR}(\varphi_1, \varphi_2) \leftrightarrow \mathbf{WPR}(\varphi_1 \wedge \neg\varphi_2, \varphi_2 \wedge \neg\varphi_1))$
(***Restricted Conjunctive Expansion***),

- $\vdash_{\mathsf{PL}} ((\varphi_1 \wedge \varphi_2) \leftrightarrow (\varphi_2 \wedge \varphi_3) \leftrightarrow (\varphi_3 \wedge \varphi_1) \leftrightarrow \bot)$
$\rightarrow (\mathbf{WPR}(\varphi_1 \vee \varphi_2, \varphi_3) \rightarrow (\mathbf{WPR}(\varphi_1, \varphi_3) \vee \mathbf{WPR}(\varphi_2, \varphi_3)))$
(***Restricted Disjunctive Distribution of Left Disjunction***),

- $\vdash_{\mathsf{PL}} ((\varphi_1 \wedge \varphi_2) \leftrightarrow (\varphi_2 \wedge \varphi_3) \leftrightarrow (\varphi_3 \wedge \varphi_1) \leftrightarrow \bot)$
$\rightarrow (\mathbf{WPR}(\varphi_1, \varphi_2 \vee \varphi_3) \rightarrow (\mathbf{WPR}(\varphi_1, \varphi_2) \vee \mathbf{WPR}(\varphi_1, \varphi_3)))$
(***Restricted Disjunctive Distribution of Right Disjunction***),

- $\vdash_{\mathsf{PL}} ((\varphi_1 \wedge \varphi_2) \leftrightarrow (\varphi_2 \wedge \varphi_3) \leftrightarrow (\varphi_3 \wedge \varphi_1) \leftrightarrow \bot)$
$\rightarrow ((\mathbf{WPR}(\varphi_1, \varphi_2) \wedge \mathbf{WPR}(\varphi_3, \varphi_2)) \rightarrow \mathbf{WPR}(\varphi_1 \vee \varphi_3, \varphi_2))$
(***Restricted Conjunctive Distribution of Left Disjunction***),

- $\vdash_{\mathsf{PL}} ((\varphi_1 \wedge \varphi_2) \leftrightarrow (\varphi_2 \wedge \varphi_3) \leftrightarrow (\varphi_3 \wedge \varphi_1) \leftrightarrow \bot)$
$\rightarrow ((\mathbf{WPR}(\varphi_1, \varphi_2) \wedge \mathbf{WPR}(\varphi_1, \varphi_3)) \rightarrow \mathbf{WPR}(\varphi_1, \varphi_2 \vee \varphi_3))$
(***Restricted Conjunctive Distribution of Right Disjunction***).

Proposition 1, together with Proposition 2 and Proposition 3, can furnish a solution to Problem 3.

4 Conclusions

We have defined the language $\mathcal{L}_{\mathsf{PL}}$ of PL, and defined a Domotor-type structured Kripke model $\mathcal{M}$ for preference that can furnish a solution to both Problem 1 and Problem 2, and provided PL with a truth definition, and provided PL with a proof system, and proved the soundness of PL in the usual way, and proved the completeness of PL by constructing the canonical model, and proved that (reflexivity), (transitivity), (connectedness) and (impartiality) are all provable in PL, and that neither (contraposition), (conjunctive expansion), (disjunctive distribution) nor (conjunctive distribution) is provable in PL, but that (restricted contraposition), (restricted conjunctive expansion), (restricted disjunctive distribution) and (restricted conjunctive distribution) are all provable in PL, which can furnish a solution to Problem 3.

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