THE RELATIONAL FOUNDATION OF MATHEMATICAL MORPHOLOGY ON HYPERGRAPHS

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The image processing operations of erosion ($\oplus$) and dilation ($\ominus$) in mathematical morphology [Ser82] are one manifestation of particular operations taking a relation, R, on a set U, and a subset $X \subseteq U$ and producing subsets $X \oplus R \subseteq U$ and $X \ominus R \subseteq U$. In modal logic, for example, the same constructions arise as $\langle \breve{R} \rangle X$ and $[R]X$ respectively where $\breve{R}$ denotes the converse of R. Mathematical morphology views these operations as ways of 'probing' an image – using a relation as an instrument to observe an image. This observation can yield a modified form of the original image in which some features are accentuated or other features removed. The algebraic foundations of mathematical morphology [HR90] are well-known and generally rely on the identification of relations on U with the lattice of join-preserving operations on the powerset $\mathscr{P}U$.

I will show how the idea of relations as ways of observing subsets can be extended from observations on sets to observations on hypergraphs. Morphology on graphs and hypergraphs has been considered before, but the effective use of relations in this area will depend on the properties of the relations which I will describe here. A relation on a hypergraph can be defined as a binary relation on the set of all nodes and all edges satisfying a condition of compatibility with the incidence structure. If $\mathcal{H}$ is a hypergraph, the relations compatible with $\mathcal{H}$ (called $\mathcal{H}$-relations) form a lattice $\mathcal{H}$-Rel. Composition of $\mathcal{H}$-relations makes $\mathcal{H}$-Rel into a quantale which is isomorphic to the quantale of join-preserving operations on the lattice of sub-graphs of $\mathcal{H}$.

Some operations on $\mathcal{H}$-relations are significantly weaker than their counterparts for relations on sets. In particular, the lattice $\mathcal{H}$-Rel does not have complements but does have both a pseudocomplement and a dual pseudocomplement and forms a bi-Heyting algebra rather than a Boolean algebra. Instead of a single converse which is an involution, we find a pair of operations which are adjoints (or, form a Galois connection). These weaker forms of converse mean that the operation of 'dilation by the converse', which forms a quantale module in the set case, gives us only a form of lax quantale module.

I will discuss how the $\mathcal{H}$-relations arise as a special case of the category-theoretic notion of profunctors or distributors, which provide a generalization of relations on sets. Relations on hypergraphs suggest several directions for further research, such as abstracting from the bi-Heyting algebra of relations together with the adjoint pair of converses and investigating the resulting generalization of relation algebra.

References

[HR90] H. J. A. M Heijmans and C. Ronse. The algebraic basis of mathematical morphology. I. Dilations and erosions. *Computer Vision Graphics and Image Processing*, 50:245–295, 1990.

[Ser82] J. Serra. *Image Analysis and Mathematical Morphology.* Academic Press, London, 1982.