VALIDITIES FOR RESIDUATED ALGEBRAS OF BINARY RELATIONS

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We will look at algebras of binary relations whose signatures Λ contain relation composition ; and its residuals \ (right) and / (left). We will also assume that an ordering ≤ is available, either as a primitive relation symbol, or defined by using the (semi)lattice operations join + or meet ·.

Terms are interpreted in an algebra C with base UC in the usual manner:

join + is union, meet · is intersection and

\[ x ; y = \{(u,v) \in U_C \times U_C : (u,w) \in x \text{ and } (w,v) \in y \text{ for some } w \} \]

\[ x \setminus y = \{(u,v) \in U_C \times U_C : \text{for every } w, (w,u) \in x \text{ implies } (w,v) \in y \} \]

\[ x / y = \{(u,v) \in U_C \times U_C : \text{for every } w, (v,w) \in y \text{ implies } (u,w) \in x \} \]

We will also need the identity constant 1′ interpreted as

\[ 1′ = \{(u,v) \in U_C \times U_C : u = v \} \]

We will look at two notions of semantics. Let τ, σ be two terms. We say that the (in)equality τ ≤ σ is (standard) valid, in symbols \[\models τ \leq σ\], if the interpretation of τ is a subset of the interpretation of σ in every algebra. On the other hand, state-semantics is defined for terms. We say that τ is state-valid, in symbols \[\models_s τ\], if 1′ ≤ τ is (standard) valid. These semantics can be restricted to special classes of algebras. In particular, we will look at commutative algebras, where \[x ; y = y ; x\] is valid. The corresponding notion of validity is denoted by using a superscript: \[\models^c\] and \[\models^c_s\].

We will consider signatures \[\{; \setminus, /\} \subseteq Λ \subseteq \{\cdot, +, ;, \setminus, /\}\] and investigate when validities \[\models\], \[\models^c\] and state-validities \[\models_s\], \[\models^c_s\] are finitely axiomatizable. That is, we look for finite set of axioms and derivation rules such that all (state-)validities can be derived. We will see that, in this respect, lower-semilattice ordered algebras generally behave better than upper-semilattice ordered algebras, although some problems are still open.

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