# On Relational Interpretation of Multimodal Categorial Logics

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**Abstract**

Several recent results show that the Lambek Calculus **L** and its close relative **L1** are sound and complete under (possibly relativized) relational interpretation. This paper transfers these results to $\mathbf{L}\diamond$, the multimodal extension of the Lambek Calculus that was proposed in Moortgat (1996). Two simple relational interpretations of $\mathbf{L}\diamond$ are proposed and shown to be sound and complete. The completeness proofs make heavy use of the method of relational labeling from Kurtonina (1995). Finally, it is demonstrated that relational interpretation provides a semantic justification for the translation from $\mathbf{L}\diamond$ to **L** from Versmissen (1996).

## 1 Introduction

In recent years several results have been obtained that show that the associative Lambek calculus **L** (Lambek 1958) and its close relative **L1** (i.e. **L** without the ban on empty premises) can be given a fairly natural relational semantics. Kurtonina (1995) showed that **L1** is sound and complete if every formula is interpreted as a binary relation over some set of states, the product operator is interpreted as relational composition, and the two directed implications $\backslash$ and $/$ as left and right residuation respectively, i.e. $\|A \backslash B\| = \overline{\|A\|^{\cup} \circ \overline{\|B\|}}$, and analogously for $\|A/B\|$. Pankrat'ev (1994) and Andréka and Mikulás (1994) prove that the same semantics is sound and complete for **L** if the interpretation of formulas is relativized to a certain transitive relation $<$ on the set of states.

Recent linguistic applications of categorial logics make heavy usage of multimodal extensions of **L** (cf. Moortgat 1997 for an overview), and it is thus an interesting question whether the mentioned results for **L** carry over to multimodal logics. The present paper contains two results pertaining to this issue. We provide two relational semantics for $\mathbf{L}\diamond$—the extension of **L** with pairs of unary residuation modalities (cf. Moortgat 1996). We establish soundness and completeness of these semantics. Finally, we point out some proof-theoretic repercussions of these model-theoretic results.

## 2 Relational semantics for the Lambek Calculus

Formulas of the Lambek Calculus **L** are defined by the closure of a set of primitive types under the three binary connectives $\bullet$, $\backslash$, and $/$. Derivability is given by the following sequent rules, where $A$, $B$ etc. range over formulas and $X, Y$ etc. over finite sequences of formulas. As an additional constraint, premises of sequents must not be empty.

Definition 1 (Sequent Calculus):

$$\frac{}{A \Rightarrow A}\,[id] \qquad \frac{X \Rightarrow A \qquad Y, A, Z \Rightarrow B}{Y, X, Z \Rightarrow B}\,[Cut]$$

$$\frac{X \Rightarrow A \qquad Y, B, Z \Rightarrow C}{Y, X, A \setminus B, Z \Rightarrow C}\,[\setminus L] \qquad \frac{A, X \Rightarrow B}{X \Rightarrow A \setminus B}\,[\setminus R]$$

$$\frac{X \Rightarrow A \qquad Y, B, Z \Rightarrow C}{Y, B/A, X, Z \Rightarrow C}\,[/L] \qquad \frac{X, A \Rightarrow B}{X \Rightarrow B/A}\,[/R]$$

$$\frac{X, A, B, Y \Rightarrow C}{X, A \bullet B, Y \Rightarrow C}\,[\bullet L] \qquad \frac{X \Rightarrow A \qquad Y \Rightarrow B}{X, Y \Rightarrow A \bullet B}\,[\bullet R]$$

In Pankrat'ev (1994) and Andréka and Mikulás (1994) it is shown that **L** is sound and complete with respect to the following semantics. Let a model consist of a set of possible worlds $W$, a transitive relation $<$ on $W$, and a valuation function $V$ that maps atomic formulas to sub-relations of $<$. The semantics of complex formulas is given by the following clauses:

Definition 2 (Relational semantics):

$$\begin{array}{rll}
\langle a,b\rangle \models p & \textit{iff} & \langle a,b\rangle \in V(p) \\
\langle a,b\rangle \models A \bullet B & \textit{iff} & a < b \wedge \exists c(\langle a,c\rangle \models A \wedge \langle c,b\rangle \models B) \\
\langle a,b\rangle \models A \setminus B & \textit{iff} & a < b \wedge \forall c(\langle c,a\rangle \models A \Rightarrow \langle c,b\rangle \models B) \\
\langle a,b\rangle \models B/A & \textit{iff} & a < b \wedge \forall c(\langle b,c\rangle \models A \Rightarrow \langle a,c\rangle \models B) \\
\langle a,b\rangle \models A, X & \textit{iff} & a < b \wedge \exists c(\langle a,c\rangle \models A \wedge \langle c,b\rangle \models X)
\end{array}$$

A sequent $A_1, \ldots, A_n \Rightarrow B$ is valid iff for all models $M$ and possible worlds $a, b$, if $\langle a,b\rangle \models A_1, \ldots, A_n$, then $\langle a,b\rangle \models B$. If we identify the relation $<$ with $W \times W$, we arrive at a notion of validity that corresponds to derivability in **L1** (which is **L** without the restriction to non-empty premises), as shown in Andréka and Mikulás (1994) and in Kurtonina (1995)—this correspondence between frame conditions and proof theoretic characterizations of the corresponding logic is akin to analogous results in the real of modal logic.

## 3 Multimodal extension

**L** can be extended to its multimodal version **L**$\Diamond$ by adding a finite family of pairs of unary connectives $\Diamond_i$ and $\Box^{\downarrow}_i$, and by extending the sequent calculus with the following rules (taken form Moortgat (1996), who proves Cut Elimination and Decidability):[1]

Definition 3 (Sequent Calculus for **L**$\Diamond$):

$$\frac{X, ({}_i A)_i, Y \Rightarrow B}{X, \Diamond_i A, Y \Rightarrow B}\,[\Diamond_i L] \qquad \frac{X \Rightarrow A}{({}_i X)_i \Rightarrow \Diamond_i A}\,[\Diamond_i R]$$

$$\frac{X, A, Y \Rightarrow B}{X, ({}_i \Box^{\downarrow}_i A)_i, Y \Rightarrow B}\,[\Box^{\downarrow}_i L] \qquad \frac{({}_i X)_i \Rightarrow A}{X \Rightarrow \Box^{\downarrow} A}\,[\Box^{\downarrow}_i R]$$

The premise of a sequent is now a bracketed sequence of formulas, i.e. a finite labeled tree. The subscript $i$ will be dropped in the remainder of the paper if no confusion arises.

[1] A note on terminology: The term "mode" is used as referring to a family of residuated operators here, not to the indices in the syntactic representation. So **L**$\Diamond$, which just comprises one family of binary and one family of unary connectives, would also qualify as "multimodal".

There are two ways how the relational semantics given above can be extended to the multimodal calculi. The first option is inspired by the way modal formulas are interpreted in Kripke semantics. If we use a procedural metaphor, to verify a formula $\Diamond A$ in a world $a$, we (i) make a transition from $a$ to some other world $b$ that is related to $a$ via the accessibility relation $R$, (ii) we verify $A$ in $b$, and (iii) we make a transition in the reverse direction back to $a$. The main novelty in a genuinely dynamic interpretation is the fact that verifying $A$ may lead us to a world $c$ that is distinct from $b$, and accordingly, making a $R^{-1}$-transition from $c$ may lead us to a world $d$ that is distinct from $a$. The static and the dynamic picture is given schematically in figure 1.

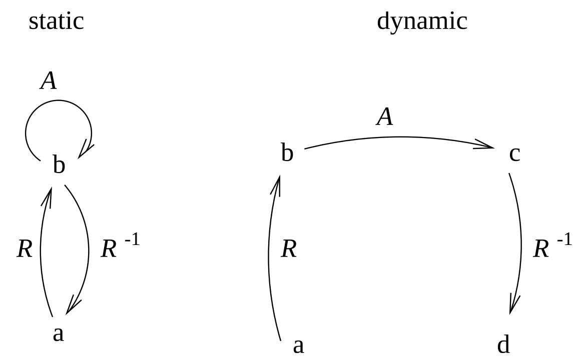


Fig. 1: Static and vertical dynamic interpretation of $\Diamond A$

Note that the input-output pairs $\langle a, d\rangle$ and $\langle b, c\rangle$ have to be related by the ordering relation $<$, while there is no such restriction for the $R$-relation. Inspired by the picture we might say that formulas relate points horizontally, while the accessibility relation $R$ is vertical. Following this suggestion, we call this semantics vertical relational semantics.

Formally, a vertical relational model for $\mathbf{L}\Diamond$ is a model for $\mathbf{L}$ enriched with a family of binary relations $R_i$ on $W$. The recursive truth definition is given below.

Definition 4 (Vertical relational Semantics for $\mathbf{L}\Diamond$):

$$
\begin{array}{rcl}
\langle a,b\rangle \models_v p & \textit{iff} & \langle a,b\rangle \in V(p)\\
\langle a,b\rangle \models_v A \bullet B & \textit{iff} & a<b \wedge \exists c(\langle a,c\rangle \models_v A \wedge \langle c,b\rangle \models_v B)\\
\langle a,b\rangle \models_v A \setminus B & \textit{iff} & a<b \wedge \forall c(\langle c,a\rangle \models_v A \Rightarrow \langle c,b\rangle \models_v B)\\
\langle a,b\rangle \models_v B/A & \textit{iff} & a<b \wedge \forall c(\langle b,c\rangle \models_v A \Rightarrow \langle a,c\rangle \models_v B)\\
\langle a,b\rangle \models_v \Diamond_i A & \textit{iff} & a<b \wedge \exists c,d(aR_ic \wedge bR_id \wedge \langle c,d\rangle \models_v A)\\
\langle a,b\rangle \models_v \Box_i^{\downarrow} A & \textit{iff} & a<b \wedge \forall c,d(cR_ia \wedge dR_ib \wedge c<d \Rightarrow \langle c,d\rangle \models_v A)\\
\langle a,b\rangle \models_v A, X & \textit{iff} & a<b \wedge \exists c(\langle a,c\rangle \models_v A \wedge \langle c,b\rangle \models_v X)\\
\langle a,b\rangle \models_v ({}_iX)_i & \textit{iff} & a<b \wedge \exists c,d(aR_ic \wedge bR_id \wedge \langle c,d\rangle \models_v X)
\end{array}
$$

We say that a sequent $X \Rightarrow A$ is vertically valid ($\models_v X \Rightarrow A$) iff for all models $M$ and worlds $a$ and $b$: if $M, \langle a,b\rangle \models_v X$, then $M, \langle a,b\rangle \models_v A$.

The second option for a relational interpretation of $\mathbf{L}\Diamond$ is inspired by the embedding from $\mathbf{L}\Diamond$ to $\mathbf{L}$ proposed in Versmissen (1996). Here $\Diamond A$ is translated as $t_0 \bullet A \bullet t_1$, where $t_0$ and $t_1$ are two fresh atomic formulas of $\mathbf{L}$. Adapted to relational semantics, this means that there are two distinguished relations $R$ and $S$ (intuitively corresponding to the formulas $t_0$ and $t_1$), and a $\Diamond A$-transition can be decomposed into an $R$-transition, followed by an $A$-step and an $S$-step (figure 2). $R$ and $S$ have to be sub-relations of $<$; thus the resulting semantics can be dubbed horizontal semantics.

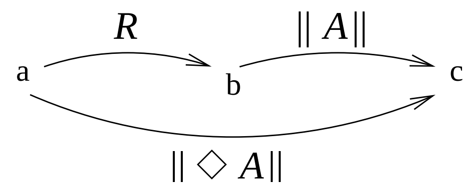


Fig. 2: Horizontal dynamic interpretation of $\Diamond A$

To make this precise, a horizontal relational model for $\mathbf{L}\Diamond$ is a model for $\mathbf{L}$ which is enriched by a family of pairs of relations $R_i$ and $S_i$ on $W$ such that for all $i$, $R_i, S_i \subseteq <$.

Definition 5 (Horizontal relational Semantics for $\mathbf{L}\Diamond$):

$$
\begin{array}{rcl}
\langle a,b\rangle \models_h p & \textit{iff} & \langle a,b\rangle \in V(p) \\
\langle a,b\rangle \models_h A \bullet B & \textit{iff} & a < b \wedge \exists c(\langle a,c\rangle \models_h A \wedge \langle c,b\rangle \models_h B) \\
\langle a,b\rangle \models_h A \setminus B & \textit{iff} & a < b \wedge \forall c(\langle c,a\rangle \models_h A \Rightarrow \langle c,b\rangle \models_h B) \\
\langle a,b\rangle \models_h B/A & \textit{iff} & a < b \wedge \forall c(\langle b,c\rangle \models_h A \Rightarrow \langle a,c\rangle \models_h B) \\
\langle a,b\rangle \models_h \Diamond_i A & \textit{iff} & a < b \wedge \exists c,d(aR_ic \wedge \langle c,d\rangle \models_h A \wedge dS_ib) \\
\langle a,b\rangle \models_h \Box_i^{\downarrow} A & \textit{iff} & a < b \wedge \forall c,d(cR_ia \wedge bS_id \wedge c < d \Rightarrow \langle c,d\rangle \models_h A) \\
\langle a,b\rangle \models_h A, X & \textit{iff} & a < b \wedge \exists c(\langle a,c\rangle \models_h A \wedge \langle c,b\rangle \models_h X) \\
\langle a,b\rangle \models_h ({}_iX)_i & \textit{iff} & a < b \wedge \exists c,d(aR_ic \wedge \langle c,d\rangle \models_h X \wedge dS_ib)
\end{array}
$$

Horizontal validity is defined analogously to vertical validity.

## 4 Weak completeness of vertical relational semantics

Both notions of validity for $\mathbf{L}\Diamond$ given in the previous section are adequate in the sense that they characterize precisely the derivable sequents. These facts are proved in this and the next section, starting with vertical interpretation.

Theorem 1 (Weak Completeness): For every sequent $X \Rightarrow A$:

$$\vdash_{L\Diamond} X \Rightarrow A \text{ iff } \models_v X \Rightarrow B$$

Soundness can easily be checked by induction on the length of derivations. The completeness proof follows largely the strategy of Kurtonina (1995) in her completeness proof for $\mathbf{L1}$ in its relational interpretation. In a first step, we augment the formulas in the sequent system with labels which reflect the truth conditions of formulas. Each formula in a sequent is labeled with a pair of labels, representing the input state and the output state of the corresponding transition. Matters are somewhat complicated by the fact that we have to distinguish horizontal and vertical transitions. To do so, we assume that labels are structured objects themselves: they consist of a *state parameter* $(u, v, w \ldots)$ and a *color index* $(r, s, t, \ldots)$. The color index is written as a subscript to the state parameter. We use letters $a, b, c, \ldots$ as metavariables over labels. The idea is that horizontal transitions only change the state parameter, while vertical transitions change both components. Brackets are treated like formulas; they are labeled with input label and output label as well. For better readability, we use "$0_i$" and "$1_i$" instead of opening and closing brackets.

Definition 6 (Labeled Sequent Calculus):

$$\frac{}{u_r v_r : A \Rightarrow u_r v_r : A}\,[id]$$

$$\frac{X \Rightarrow ab : A \qquad Y, ab : A, Z \Rightarrow cd : B}{Y, X, Z \Rightarrow cd : B}\,[Cut]$$

$$\frac{X \Rightarrow ab : A \qquad Y, ac : B, Z \Rightarrow de : C}{Y, X, bc : A \setminus B, Z \Rightarrow de : C}\,[\setminus L]$$

$$\frac{\underline{u_r} v_r : A, X \Rightarrow \underline{u_r} w_r : B}{X \Rightarrow v_r w_r : A \setminus B}\,[\setminus R]$$

$$\frac{X \Rightarrow ab : A \qquad Y, cb : B, Z \Rightarrow de : C}{Y, ac : B/A, X, Z \Rightarrow de : C}\,[/L]$$

$$\frac{X, u_r \underline{v_r} : A \Rightarrow w_r \underline{v_r} : B}{X \Rightarrow w_r u_r : B/A}\,[/R]$$

$$\frac{X, u_r \underline{v_r} : A, \underline{v_r} w_r : B, Y \Rightarrow de : C}{X, u_r w_r : A \bullet B, Y \Rightarrow de : C}\,[\bullet L]$$

$$\frac{X \Rightarrow ab : A \qquad Y \Rightarrow bc : B}{X, Y \Rightarrow ac : A \bullet B}\,[\bullet R]$$

$$\frac{X, u_r \underline{v_s} : 0_i, \underline{v_s}\underline{w_s} : A, \underline{w_s} x_r : 1_i, Y \Rightarrow ef : B}{X, u_r x_r : \Diamond_i A, Y \Rightarrow ef : B}\,[\Diamond_i L]$$

$$\frac{X \Rightarrow u_r v_r : A}{w_s u_r : 0_i, X, v_r x_s : 1_i \Rightarrow w_s x_s : \Diamond_i A}\,[\Diamond_i R]$$

$$\frac{X, u_r v_r : A, Y \Rightarrow ab : B}{X, u_r w_{\underline{s}} : 0_i, w_{\underline{s}} x_{\underline{s}} : \Box_i^{\downarrow} A, x_{\underline{s}} v_r : 1_i, Y \Rightarrow ab : B}\,[\Box_i^{\downarrow} L]$$

$$\frac{\underline{u_r} v_s : 0_i, X, w_s \underline{x_r} : 1_i \Rightarrow \underline{u_r}\underline{x_r} : A}{X \Rightarrow v_s w_s : \Box^{\downarrow} A}\,[\Box_i^{\downarrow} R]$$

The underlined labels have to be fresh, i.e. they must not occur elsewhere in the sequent.

Definition 7 (Proper and canonical labeling): *A sequent $a_1 b_1 : A_1, \ldots, a_n b_n : A_n \Rightarrow ab : A$ is properly labeled iff*

- $a_1 = a, b_n = b$
- $\forall i (1 \leq i < n \rightarrow b_i = a_{i+1})$.
- *If $A_i = 0$ or $A_i = 1$, $a_i$ and $b_i$ have different colors.*
- *Otherwise, $a_i$ and $b_i$ have the same color.*
- *If $A_i = 0$, then there is a $j > i$ with $A_j = 1$ and the input color of $A_i$ equals the output color of $A_j$ and vice versa.*
- *If $A_i = 1$, then there is a $j < i$ with $A_j = 0$ and the input color of $A_i$ equals the output color of $A_j$ and vice versa.*

*It is canonically labeled iff*

- *it is properly labeled.*
- *Each label occurs exactly twice.*

Lemma 1: *If a sequent is derivable, it is properly labeled.*

**Proof:**
By induction over the length of derivations. $\dashv$

Lemma 2 (Renaming Lemma): *If $a_0a_1 : A_1, \ldots, a_{n-1}a_n : A_n \Rightarrow a_0a_n : B$ is derivable, then the result of renaming all occurrences of an arbitrary $a_i$ with a label of the same color is also derivable.*

**Proof:**
By induction on the length of derivations. $\dashv$

The idea of the completeness proof can be sketched as follows. Suppose a given sequent $A \Rightarrow B$ is underivable.[2] Then the labeled sequent $ab : A \Rightarrow ab : B$ ($a$ and $b$ being distinct and having the same color) is underivable as well (otherwise we could transform every proof of the latter into a proof of the former simply by dropping the labels). We will construct a falsifying model whose domain is the set of labels and which has the property that $\langle a, b\rangle \models A, \langle a, b\rangle \not\models B$. To this end, we mark labeled formulas with their intended truth value. This gives us the set $\{Tab : A, Fab : B\}$. Let us call such sets T–F sets. We show that every consistent T–F set can be extended to a maximally consistent T–F set, and furthermore that each maximally consistent T–F set corresponds to a model which verifies all T-marked and falsifies all F-marked formulas in it. Hence for each underivable sequent we can construct a falsifying model, which means that every valid sequent is derivable.

To simplify the model construction, we reify the ordering relation and treat $<$ as a formula too.

Definition 8 (T–F set): *A T–F formula is either a formula of $\boldsymbol{L}\diamond$, "0", "1", or "$<$", which is labeled with a pair of labels and marked either with "T" or with "F". A T–F set is a set of T–F formulas.*

By $\sqsubset_\Delta$ we refer to the transitive closure of the relation $\{\langle a, b\rangle | Tab :<\in \Delta\}$.

Definition 9 (Maxiconsistency): *A T–F set $\Delta$ is called* maxiconsistent *if it obeys the following constraints:*

- *For any labeled formula $ab : A$ ($A \neq 0, 1, <$), either $Tab : A$ or $Fab : A$ is in $\Delta$, but not both.*
- *If $Tab : A \in \Delta$ and $a \neq 0, 1$, then $Tab :<\in \Delta$.*
- *$\Delta$ is* saturated, *i.e.*
  - *(i) If $Fab : A \setminus B \in \Delta$ and $a \sqsubset_\Delta b$, then there is a $c$ such that $Tca : A, Fcb : B \in \Delta$.*
  - *(ii) If $Fab : A/B \in \Delta$ and $a \sqsubset_\Delta b$, then there is a $c$ such that $Tbc : B, Fac : A \in \Delta$.*
  - *(iii) If $Tab : A \bullet B \in \Delta$, then there is a $c$ such that $Tac : A, Tcb : B \in \Delta$.*
  - *(iv) If $Tab : \diamond A \in \Delta$, then there are $c$ and $d$ such that $Tac : 0, Tcd : A, Tdb : 1 \in \Delta$.*
  - *(v) If $Fab : \Box^{\downarrow} A \in \Delta$ and $a \sqsubset_\Delta b$, then there are $c$ and $d$ such that $Tca : 0, Fcd : A, Tbd : 1, Tcd :<\in \Delta$.*
  - *(vi) $Tab : 0 \in \Delta$ iff $Tba : 1 \in \Delta$.*

[2] It is sufficient to show completeness for sequents with a single formula as premise, since any proper sequent can be transformed into a formula with the same truth conditions by replacing commas with products and bracket pairs with diamonds.

- $\Delta$ *is deductively closed, i.e. if a sequent* $\alpha_1, \dots, \alpha_n \Rightarrow \beta$ *derivable, and for all* $1 \leq i \leq n$ : $T\alpha_i \in \Delta$, *then* $T\beta \in \Delta$.

From a maxiconsistent set we can construct a model in the following way:

Definition 10 (Canonical Model): *Let* $\Delta$ *be a maxiconsistent set. The canonical model for* $\Delta$ *is* $M_\Delta = \langle W, <, I, \{R_i | i \in I\}, V\rangle$, *where*

1. $W$ *is the set of labels occurring in* $\Delta$.
2. $a < b$ *iff* $a \sqsubset_\Delta b$
3. $aR_ib$ *iff* $Tab : 0_i \in \Delta$
4. $\langle a, b\rangle \in V(p)$ *iff* $Tab : p \in \Delta$.

Fact 1: *If* $\Delta$ *is maxiconsistent,* $M_\Delta$ *is a vertical relational model for* $\boldsymbol{L}\Diamond$

**Proof:**
Transitivity of $<$ follows immediately from the model construction. The requirement that $\Delta$ is maxiconsistent ensures that $V(p) \subseteq <$ for arbitrary atoms $p$. $\dashv$

Lemma 3 (Truth Lemma): *For all maxiconsistent sets* $\Delta$, *formulas* $A$ *and labels* $a, b$:

$$Tab : A \in \Delta \textit{ iff } M_\Delta, ab \models_h A$$

**Proof:**
By induction on the complexity of $A$. For the base case, the conclusion follows from the definition of $M_\Delta$.

1. $A = B \bullet C, \Rightarrow$ Since $\Delta$ is saturated, there is a $c$ such that $Tac : B, Tcb : C \in \Delta$. By induction hypothesis, $ac \models B, cb \models C$, and furthermore $a < b$, hence $ab \models B \bullet C$.
2. $\Leftarrow$ By the semantics of $\bullet$, there is a $c$ such that $ac \models B, cb \models C$. By induction hypothesis $Tac : B, Tcb : C \in \Delta$. Since $ac : B, cb : C \Rightarrow ab : B \bullet C$, deductive closure of $\Delta$ gives us $Tab : B \bullet C \in \Delta$.
3. $A = B \setminus C, \Rightarrow$ Suppose $ab \not\models B \setminus C$. Since $a < b$ by maxiconsistency, there is a $c$ such that $ca \models B, cb \not\models C$. By induction hypothesis, $Tca : B, Fcb : C \in \Delta$. Since $ca : B, ab : B \setminus C \Rightarrow cb : C$, $Tcb : C \in \Delta$, which violates consistency of $\Delta$.
4. $\Leftarrow$ Suppose $Tab : B \setminus C \notin \Delta$. By completeness of $\Delta$, $Fab : B \setminus C \in \Delta$. Since $a < b$ by the semantics of "$\setminus$", $a \sqsubset_\Delta b$ and therefore saturation entails that there is a $c$ such that $Tca : B, Fcb : C \in \Delta$. By induction hypothesis, $ca \models B, cb \not\models C$, which is impossible.
5. $A = B/C$ Likewise.
6. $A = \Diamond B, \Rightarrow$ By saturation, $Tab :<\in \Delta$, and there are $c$ and $d$ such that $Tac : 0, Tcd : B, Tdb : 1 \in \Delta$. By induction hypothesis, $cd \models B$. The construction of $M_\Delta$ ensures that $aRc$, $bRd$, and $a < b$. Hence $ab \models \Diamond B$.
7. $\Leftarrow$ By the semantics of $\Diamond$, there are $c$ and $d$ such that $aRc, bRd$, and $cd \models B$. By induction hypothesis, $Tcd : B \in \Delta$. By the construction of $M_\Delta$ and maxiconsistency, $Tac : 0, Tdb : 1 \in \Delta$. Since $\vdash ac : 0, cd : B, db : 1 \Rightarrow ab : \Diamond B$ and $\Delta$ is deductively closed, $Tab : \Diamond B \in \Delta$.

8. $A = \Box^{\downarrow} B, \Rightarrow$ Suppose $ab \not\models \Box^{\downarrow} B$. Then there are $c$ and $d$ such that $cRa, dRb$, $c < d$, and $cd \not\models B$. By induction hypothesis, $Fcd : B \in \Delta$, and the construction of $M_\Delta$ ensures that $Tca : 0, Tbd : 1 \in \Delta$. Since $\vdash ca : 0, ab : \Box^{\downarrow} B, bd : 1 \Rightarrow cd : B$, $Tcd : b \in \Delta$, which violates consistency.

9. $\Leftarrow$ Suppose $Tab : \Box^{\downarrow} B \notin \Delta$. By completeness, $Fab : \Box^{\downarrow} B \in \Delta$. By saturation, there are $c$ and $d$ such that $Tca : 0, Tbd : 1, c \sqsubset_\Delta d, Fcd : B \in \Delta$. Hence $cRa, dRb, c < d$ and $cd \not\models B$, which is impossible according to the truth conditions for "$\Box^{\downarrow}$". $\dashv$

To extend the initial T–F set to a saturated one, we constructively enforce saturation by adding "Henkin witnesses":

Assume an ordering of the set of labels.

Definition 11 (Henkin witnesses): *Let $\Delta$ be a T–F set and $\alpha$ be a T–F labeled formula. $a$ and $b$ are always assumed to be distinct.*

*(i) If $\alpha = Tab : A \bullet B$, then $H(\Delta, \alpha) = \Delta \cup \{\alpha, Tac : A, Tac :<, Tcb : B, Tcb :<\}$, where $c$ is the first label having the same color as $a$ which does not occur in $\Delta$.*

*(ii) If $\alpha = Fab : A \setminus B$ and $a \sqsubset_\Delta b$, then $H(\Delta, \alpha) = \Delta \cup \{\alpha, Tca : A, Tca :<, Fcb : B\}$, where $c$ is the first label of $a$'s color not occurring in $\Delta$.*

*(iii) If $\alpha = Fab : A/B$ and $a \sqsubset_\Delta b$, then $H(\Delta, \alpha) = \Delta \cup \{\alpha, Tbc : B, Tbc :<, Fac : A\}$, where $c$ is the first label of $a$'s color not occurring in $\Delta$.*

*(iv) If $\alpha = Tab : \Diamond A$, then $H(\Delta, \alpha) = \Delta \cup \{\alpha, Taw_r : 0, Tw_r a : 1, Tw_r u_r : A, Tw_r u_r :<, Tu_r b : 1, Tbu_r : 0\}$, where $w$ and $u$ are the first distinct state parameters and $r$ is the first color index not occurring in $\Delta$.*

*(v) If $\alpha = Fab : \Box^{\downarrow} A$ and $a \sqsubset_\Delta b$, then $H(\Delta, \alpha) = \Delta \cup \{\alpha, Tw_r a : 0, Taw_r : 1, Fw_r u_r : A, Tbu_r : 1, Tu_r b : 0, Tw_r u_r :<\}$ where $w$ and $u$ are the first distinct state parameters and $r$ is the first color index not occurring in $\Delta$.*

*(vi) Else $H(\Delta, \alpha) = \Delta$.*

Adding Henkin witnesses preserves three properties of T–F sets that are essential to prove maxi-consistency.

Definition 12 (Deep Consistency): *A set $\Delta$ is called* deeply consistent *iff it has the properties that if $\vdash \alpha_1, \ldots, \alpha_n \Rightarrow \beta$ and $T\alpha_i \in \Delta$ for all $1 \leq i \leq n$, then $F\beta \notin \Delta$.*

Definition 13 (Acyclicity): *A T–F set $\Delta$ is called* acyclic *iff there is no sequence of labels $a_1, \ldots, a_n$ such that $Ta_{i-1}a_i :<, Ta_n a_1 :<\in \Delta$.*

Definition 14 (Well-Coloredness): *A T–F set $\Delta$ is well-colored iff the following conditions hold:*

- *If $Tab :<\in \Delta$, then $a$ and $b$ have the same color.*
- *If $Tab : 0 \in \Delta$ or $Tab : 1 \in \Delta$, then $a$ and $b$ have different colors.*

Lemma 4: *If $\alpha \in \Delta$ and $\Delta$ is deeply consistent, acyclic and well-colored, then $H(\Delta, \alpha)$ is also deeply consistent, acyclic and well-colored.*

**Proof:**
As for acyclicity, observe that addition of $Tac :<$ cannot destroy it provided $c$ is fresh and $a \neq c$. This covers cases (ii) trough (v). In the first cases, assume that adding $Tac :<$, $Tcb :<$ destroys acyclicity. This means that there is a sequence $a_1, \ldots, a_n$ such that $Ta_{i-1}a_i :<$, $Ta_na_1 :< \in \Delta \cup \{Tac :<, Tcb :<\}$. In this sequence, all occurrences of $c$ have to occur between $a$ and $b$. Since the fact that $Tab : A \bullet B \in \Delta$ entails that $Tab :< \in \Delta$, removing all occurrences of $c$ would yield a closed cycle for $\Delta$, contra assumption.

Preservation of well-coloredness is immediate from the definition of Henkin witnesses.

To prove preservation of deep consistency, we assume the contrary and derive a contradiction in each case.

(i) Since in every derivable sequent each label occurs an even number of times, the sequent that violates deep consistency must have the form $X_1, ac : A, cb : B, \ldots, X_n, ac : A, cb : B, Y \Rightarrow \alpha$ where all formulas occurring in $X_1, \ldots, X_n, Y, \alpha$ already occur in $\Delta$. By the renaming lemma, thence the following sequent is also valid: $X_1, ac_1 : A, c_1b : B, \ldots, X_n, ac_n : A, c_nb : B, Y \Rightarrow \alpha$, from which we can derive $X_1, ab : A \bullet B, \ldots, X_n, ab : A \bullet B, Y \Rightarrow \alpha$ Since all formulas involved are already in $\Delta$ and $\Delta$ is deeply consistent, $F\alpha$ cannot be in $\Delta$, which is a contradiction.

(ii) By the same reasoning as above, both new formulas must occur in the sequent that causes violation of deep consistency. Hence its conclusion is $cb : B$. The only place where the other occurrence of $c$ can possibly occur is the first premise, hence the sequent has the form $ca : A, X \Rightarrow cb : B$ with $X$ consisting only of old T-marked formulas. Since $a \sqsubset_\Delta b$ and $\Delta$ is acyclic and hence irreflexive, $a \neq b$ which ensures that $X$ is non-empty. Therefore from this sequent we can derive $X \Rightarrow ab : A \setminus B$, which is excluded by the deep consistency of $\Delta$.

(iii) Likewise.

(iv) Suppose $w_ra : 1$ occurs in the sequent that destroys deep consistency. Since $w_r$ is fresh, there is no F-formula with $w_r$ as input label, and the only T-formula with $w_r$ as output label is $Taw_r : 0$. Hence the sequent in question would have the form $X, aw_r : 0, w_ra : 1, Y \Rightarrow \alpha$, which is impossible since there are no valid sequents where a closing bracket immediately follows an opening bracket. In the same way it can be shown that $Tu_rb : 0$ cannot be involved in the destruction of deep consistency. Thus by familiar reasoning, the guilty sequent has the form $X_1, aw_r : 0, w_ru_r : A, u_rb : 1, \ldots, X_n, aw_r : 0, w_ru_r : A, u_rb : 1, Y \Rightarrow \alpha$. By the renaming lemma, $X_1, aw_{r,1} : 0, w_{r,1}u_{r,1} : A, u_{r,1}b : 1, \ldots, X_n, aw_{r,n} : 0, w_{r,n}u_{r,n} : A, u_{r,n}b : 1, Y \Rightarrow \alpha$ with $w_{r,i}$ and $u_{r,i}$ fresh is also valid. From this we derive the validity of $X_1, ab : \Diamond A, \ldots, ab : \Diamond A, Y \Rightarrow \alpha$ which is incompatible with the assumption of the deep consistency of $\Delta$.

(v) Suppose $aw_r : 1$ would occur in the sequent that undermines deep consistency. Since every valid sequent is properly labeled and $w_r$ is a new label, this sequent has to take the form $A_1, \ldots, aw_r : 1, w_ra : 0, \ldots A_n \Rightarrow \alpha$, where all premises are T-marked and the conclusion is F-marked in $H(\Delta, \alpha)$. By proper labeling we know that $aw_r : 1$ has to be preceded by $cu_r : 0$ for some $c, u$. But this is impossible since $r$ is a new color. Thus $Taw_r : 1$ cannot destroy deep consistency. The same case can be made for $Tu_rb : 0$. Therefore destruction of deep consistency entails that there is a valid sequent $w_ra : 0, X, bu_r : 1 \Rightarrow w_ru_r : A$ such that all formulas in $X$ are T-marked in $\Delta$. Since $Tab :< \in \Delta$, $a \neq b$ due to acyclicity and hence $X$ is non-empty. Therefore the sequent $x \Rightarrow ab : \Box^{\downarrow} A$ is also valid, which contradicts deep consistency of $\Delta$.

(vi) Immediate. $\dashv$

It remains to be shown that any deeply consistent, acyclic and well-colored T–F set can be extended to a maxiconsistent T–F set.

Lemma 5: *If $\Delta$ is deeply consistent, acyclic, and well-colored, $A \neq 0, 1$ and $a$ and $b$ have the same color, then either $\Delta \cup \{Tab : A, Tab :<\}$ or $\Delta \cup \{Fab : A\}$ is deeply consistent, acyclic, and well-colored.*

**Proof:**
Suppose adding $Fab : A$ destroys deep consistency, acyclicity, or well-coloredness. Adding an F-marked formula cannot destroy acyclicity or well-coloredness, hence $\Delta \cup \{Fab : A\}$ is not deeply consistent. This means that there is a set of formulas $Tac_1 : A_1, \ldots, Tc_{n-1}b : A_n \in \Delta$ such that $ac_1 : A_1, \ldots, c_{n-1}b : A_n \Rightarrow ab : A$ is valid. Now suppose adding $Tab : A$ would destroy deep consistency, too. Then there would be a valid sequent $X_1, ab : A, \ldots, X_m, ab : A, Y \Rightarrow cd : C$ such that $Fcd : C \in \Delta$ and $X_1, \ldots, X_m$ consist of T-marked formulas from $\Delta$. By repeated application of Cut we would obtain the valid sequent $X_1, ac_1 : A_1, \ldots, c_{n-1}b : A_n, \ldots, X_n, ac_1 : A_1, \ldots, c_{n-1}b : A_n, Y \Rightarrow cd : C$, where the premise consists only of T-marked formulas and the conclusion is F-marked, which is excluded by the deep consistency of $\Delta$. Adding $Tab :<$ cannot destroy acyclicity since $Tac_1 :<, \ldots, Tc_{n-1}b :<$ are in $\Delta$ and $\Delta$ is acyclic. Preservation of well-coloredness is obvious. ⊣

This allows us to construct a maxiconsistent set by the following procedure:

Definition 15: *Let $\Delta$ be a deeply consistent set and $\varphi$ be an enumeration of labeled formulas (excluding 0, 1, and $<$).*

1. $\Delta_0 = \Delta$
2. *If $\varphi_n = ab : A$, and $\Delta_n \cup \{T\varphi_n, Tab :<\}$ is deeply consistent, acyclic, and well-colored, then* $\Delta_{n+1} = H(\Delta_n \cup \{T\varphi_n, Tab :<\}, T\varphi_n)$.
3. *Otherwise* $\Delta_{n+1} = H(\Delta_n \cup \{F\varphi_n\}, F\varphi_n)$.
4. $\Delta_\omega = \bigcup_{n\in\omega} \Delta_n$.

Lemma 6: *If $n < m$, $a$ and $b$ are labels occurring in $\Delta_n$, and $\neg a \sqsubset_{\Delta_n} b$, then $\neg a \sqsubset_{\Delta_m} b$.*

**Proof:**
Induction over $n$ and $m$. ⊣

Lemma 7: *If $\Delta$ is deeply consistent, acyclic, and well-colored, and $\forall a, b(Tab : 0 \in \Delta \leftrightarrow Tba : 1 \in \Delta)$, then $\Delta_\omega$ is maxiconsistent.*

**Proof:**
By the construction, either $T\alpha$ or $F\alpha$ is in $\Delta_\omega$ for all labeled formulas $\alpha$. Lemmas 4 and 5 ensure that each $\Delta_n$ is deeply consistent. If both $T\alpha$ and $F\alpha$ were in $\Delta_\omega$, they would be in some $\Delta_n$ too, which is impossible since these are deeply consistent. An inspection of the clauses for Henkin witnesses shows that each addition of a formula $Tab : A$ is accompanied by addition of $Tab :<$. Clauses (i) – (v) of saturation are ensured by closure under Henkin witnesses together with lemma 6. By assumption, clause (vi) of the definition of saturation hold of $\Delta_0$, and it is easy to see that it is preserved under every step from $\Delta_n$ to $\Delta_{n+1}$. Thus it also holds of $\Delta_\omega$ since otherwise it would already fail for some $\Delta_n$. Since $\Delta_\omega$ is complete, failure of deductive closure would entail failure of deep consistency for some $\Delta_n$. ⊣

Lemma 8: *If $a_1b_1 : A_1, \ldots, a_nb_n : A_n \Rightarrow \alpha$ is canonically labeled and underivable, then $\{Ta_ib_i : A_i, F\alpha\} \cup \{Ta_ib_i :< | 0 \neq A_i \neq 1\}$ is deeply consistent, acyclic, and well-colored.*

**Proof:**
Since the sequent is canonically labeled, the only properly labeled sequent made from its components is the original sequent itself. Hence there is no valid sequent consisting only of formulas from the set in question. Acyclicity and well-coloredness follow from the definition of canonical labeling. $\dashv$

Lemma 9: *If $ab : A \Rightarrow ab : B$ is derivable in the labeled calculus, $A \Rightarrow B$ is derivable in the unlabeled calculus.*

**Proof:**
Simply drop the labels in the proof, and replace "0" by "(" and "1" by ")". $\dashv$

Now suppose $A \Rightarrow B$ is underivable in the unlabeled calculus. By the last lemma, $w_r u_r : A \Rightarrow w_r u_r : B$ ($w$ and $u$ distinct) is canonically labeled and underivable in the labeled calculus. Hence in the canonical model constructed from $\{Tw_r u_r : A, Tw_r u_r :<, Fw_r u_r : B\}_\omega$, $\langle w_r, u_r\rangle$ verifies $A$ and falsifies $B$. This completes the proof of Theorem 1. $\dashv$

## 5 Weak Completeness of horizontal relational semantics

Theorem 2 (Weak Completeness): For every sequent $X \Rightarrow A$:

$$\vdash_{L\diamond} X \Rightarrow A \text{ iff } \models_h X \Rightarrow B$$

The soundness proof is again a straightforward induction over the length of derivations. The completeness proof is very similar to the proof in the previous section, so I will content myself with pointing out the differences.

Definition 16:
*Let $\Delta$ be a $T$–$F$ set. We say that $a \sqsubset_\Delta b$ iff there are labels $c_1, \ldots, c_n$ such that $a = c_1, b = c_n, Ta_{i-1}a_i :<\in \Delta \vee Ta_{i-1}a_i : 0 \in \Delta \vee Ta_{i-1}a_i : 1 \in \Delta$ for all $1 \leq i \leq n$.*

The definition of a maxiconsistent set now runs as follows:

Definition 17 (Maxiconsistency): *A $T$–$F$ set $\Delta$ is called* maxiconsistent *iff it obeys the following constraints:*

- *For any labeled formula $ab : A$ ($A \neq 0, 1, <$), either $Tab : A$ or $Fab : A$ is in $\Delta$, but not both.*
- *If $Tab : A \in \Delta$ and $A \neq 0, 1$, then $Tab :<\in \Delta$.*
- *$\Delta$ is* saturated, *i.e.*
  - *(i) If $Fab : A \setminus B \in \Delta$ and $a \sqsubset_\Delta b$, then there is a $c$ such that $Tca : A, Fcb : B \in \Delta$.*
  - *(ii) If $Fab : A/B \in \Delta$ and $a \sqsubset_\Delta b$, then there is a $c$ such that $Tbc : B, Fac : A \in \Delta$.*
  - *(iii) If $Tab : A \bullet B \in \Delta$, then there is a $c$ such that $Tac : A, Tcb : B \in \Delta$.*
  - *(iv) If $Tab : \diamond A \in \Delta$, then there are $c$ and $d$ such that $Tac : 0, Tcd : A, Tdb : 1 \in \Delta$.*
  - *(v) If $Fab : \Box^{\downarrow} A \in \Delta$, then there are $c$ and $d$ such that $Tca : 0, Fcd : A, Tbd : 1, Tcd :<\in \Delta$.*
  - *(vi) If $Tab : A \in \Delta$, $A, B \neq 0, 1$, then $Tab :<\in \Delta$.*
- *$\Delta$ is deductively closed, i.e. if a sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ derivable, and for all $1 \leq i \leq n$ : $T\alpha_i \in \Delta$, then $T\beta \in \Delta$.*

From a maxiconsistent set we can construct a canonical model for horizontal semantics:

Definition 18 (Canonical Model): *Let $\Delta$ be a maxiconsistent set. The canonical model for $\Delta$ is $M_\Delta = \langle W, <, I, \{R_i | i \in I\}, \{S_i | i \in I\}, V\rangle$, where*

1. *$W$ is the set of labels occurring in $\Delta$.*
2. *$a < b$ iff $a \sqsubset_\Delta b$*
3. *$aR_ib$ iff $Tab : 0_i \in \Delta$*
4. *$aS_ib$ iff $Tab : 1_i \in \Delta$*
5. *$\langle a, b\rangle \in V(p)$ iff $Tab : p \in \Delta$.*

Fact 2: *If $\Delta$ is maxiconsistent, $M_\Delta$ is a horizontal relational model for $\boldsymbol{L}\Diamond$*

**Proof:**
By the definition of $\sqsubset_\Delta$, $<$ is transitive and $R_i, S_i \subseteq <$. The requirement that $\Delta$ is maxiconsistent ensures that $V(p) \subseteq <$ for arbitrary atoms $p$. $\dashv$

Lemma 10 (Truth Lemma): *For all maxiconsistent sets $\Delta$, formulas $A$ and labels $a, b$:*

$$Tab : A \in \Delta \text{ iff } M_\Delta, ab \models A$$

**Proof:**
By induction over the complexity of $A$. Cases 1–5 are identical to the proof for vertical semantics.

7. $A = \Diamond B, \Rightarrow$ By saturation, $Tab :<\in \Delta$, and there are $c$ and $d$ such that $Tac : 0, Tcd : B, Tdb : 1 \in \Delta$. By induction hypothesis, $cd \models B$. The construction of $M_\Delta$ ensures that $aRc$, $dSb$, and $a < b$. Hence $ab \models \Diamond B$.

8. $\Leftarrow$ By the semantics of $\Diamond$, there are $c$ and $d$ such that $aRc, dSb$, and $cd \models B$. By induction hypothesis, $Tcd : B \in \Delta$. By the construction of $M_\Delta$, $Tac : 0, Tdb : 1 \in \Delta$. Since $\vdash ac : 0, cd : B, db : 1 \Rightarrow ab : \Diamond B$ and $\Delta$ is deductively closed, $Tab : \Diamond B \in \Delta$.

9. $A = \Box^\downarrow B, \Rightarrow$ Suppose $ab \not\models \Box^\downarrow B$. Then there are $c$ and $d$ such that $cRa, bSd$, $c < d$, and $cd \not\models B$. By induction hypothesis, $Fcd : B \in \Delta$, and the construction of $M_\Delta$ ensures that $Tca : 0, Tbd : 1 \in \Delta$. Since $\vdash ca : 0, ab : \Box^\downarrow B, bd : 1 \Rightarrow cd : B$, $Tcd : B \in \Delta$, which violates consistency.

10. $\Leftarrow$ Suppose $Tab : \Box^\downarrow B \notin \Delta$. By completeness, $Fab : \Box^\downarrow B \in \Delta$. By saturation, there are $c$ and $d$ such that $Tca : 0, Tbd : 1, Tcd :<, Fcd : B \in \Delta$. Hence $cRa, bSd, c < d$ and $cd \not\models B$, which is impossible due to the truth conditions for "$\Box^\downarrow$". $\dashv$

In the definition of Henkin witnesses, the clauses for the modal formulas are modified:

Definition 19 (Henkin witnesses):

*(v)* *If $\alpha = Tab : \Diamond A$, then $H(\Delta, \alpha) = \Delta \cup \{\alpha, Tac : 0, Tcd : A, Tcd :<, Tdb : 1\}$, where $c$ and $d$ are the first distinct labels not occurring in $\Delta$.*

*(vi)* *If $\alpha = Fab : \Box^\downarrow A$ and $a \sqsubset_\Delta b$, then $H(\Delta, \alpha) = \Delta \cup \{\alpha, Tca : 0, Fcd : A, Tbd : 1, Tcd :<\}$, where $c$ and $d$ are the first distinct labels not occurring in $\Delta$.*

For horizontal semantics, we can ignore well-coloredness.

Lemma 11: *If $\alpha \in \Delta$ and $\Delta$ is deeply consistent and acyclic, then $H(\Delta, \alpha)$ is also deeply consistent and acyclic.*

**Proof:**
Preservation of acyclicity is as above. As for deep consistency, the proof runs basically as above too. For the Lambek connectives, it is just identical, and for the modal operators, it is even simpler since fewer formulas are added at each step of adding Henkin witnesses.

Lemma 12: *If $\Delta$ is deeply consistent and acyclic, and $A \neq 0, 1$, then either $\Delta \cup \{Tab : A, Tab :<\}$ or $\Delta \cup \{Fab : A\}$ is deeply consistent and acyclic.*

**Proof:**
As above.

The construction of a maxiconsistent T–F set doesn't differ from the vertical case.

Lemma 13: *If $\Delta$ is deeply consistent and acyclic, then $\Delta_\omega$ is maxiconsistent.*

**Proof:**
See above.

Lemma 14: *If $a_1b_1 : A_1, \ldots, a_nb_n : A_n \Rightarrow \alpha$ is canonically labeled and underivable, then $\{Ta_ib_i : A_i, F\alpha\} \cup \{Ta_ib_i :< |0 \neq A_i \neq 1\}$ is deeply consistent and acyclic.*

**Proof:**
See above.

As in the horizontal case, the last lemma ensures that for each underivable sequent, we can construct a model that falsifies it. $\dashv$

## 6 Strong completeness

Kurtonina (1995) shows that **L1** is also complete in its relational interpretation if conceived as an "axiomatic-sequent" calculus. Under this perspective, derivability and entailment are relations between (sets of) sequents and not formulas.

Definition 20 (Derivability): *A sequent $\varphi$ is $\boldsymbol{L}\diamond$-derivable from a set of sequents $\Gamma$ iff there is a sequence of sequents $\delta_1, \ldots, \delta_n$ with $\delta_n = \varphi$ such that each $\delta_i$ is either an axiom of $\boldsymbol{L}\diamond$, an element of $\Gamma$, or it can be obtained from $\delta_1, \ldots, \delta_{i-1}$ by inference rules of $\boldsymbol{L}\diamond$.*

A sequent $X \Rightarrow A$ is said to be true in a model $M$ iff $\|X\|_M \subseteq \|A\|_M$. This leads immediately to a notion of entailments between sequents.

Definition 21 (Entailment): *A sequent $\varphi$ is (horizontally/vertically) entailed by a set of sequents $\Gamma$ iff in all models where all elements of $\Gamma$ are (horizontally/vertically) true, $\varphi$ is true as well.*

Theorem 3 (Strong Completeness): *A sequent $\varphi$ is $\boldsymbol{L}\diamond$-derivable from a set of sequents $\Gamma$ iff it is vertically entailed by $\Gamma$ iff it is horizontally entailed by $\Gamma$.*

**Proof:**
Soundness is straightforward by induction on the length of derivations. As for completeness,

Kurtonina's (1995) proof for the corresponding theorem for **L1** immediately carries over to **L**$\diamond$. We assume that $\varphi$ is not derivable from $\Gamma$ and show that it cannot be entailed. First we define the set $\Gamma_l$ as the set of all canonically labeled instances of elements of $\Gamma$. The notion of derivability of sequents above (definition 20) is extended to labeled sequents by replacing **L**$\diamond$ with its labeled version. A set $\Delta$ of labeled $T-F$ formulas is called (vertically/horizontally) $\Gamma$-maxiconsistent iff it is (vertically/horizontally) maxiconsistent and furthermore it is $\Gamma$-closed, i.e. if a sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ is derivable from $\Gamma_l$, and for all $1 \leq i \leq n : T\alpha_i \in \Delta$, then $T\beta \in \Delta$. Since $\Gamma$-maxiconsistency is a stronger notion than maxiconsistency, fact 1/2 and lemma 3/10 also hold if we replace the latter by the former. In a similar fashion, we strengthen the notion of deep consistency to $\Gamma$-consistency by replacing derivability with derivability from $\Gamma_l$. The lemmas 4–7/11–13 remain valid if we replace deep consistency with $\Gamma$-consistency. Now suppose $\Gamma \not\vdash_{\mathbf{L}\diamond} \varphi = A_1, \ldots, A_n \Rightarrow B$. Since this sequent is not derivable from $\Gamma$, neither is any of its canonically labeled versions $ab_1 : A_1, \ldots, b_{n-1}c : A_n \Rightarrow ac : B$ derivable from $\Gamma_l$. Hence $\{Tab_1 : A_1, \ldots T : b_{n-1}c : A_n, Fac : B\}$ is $\Gamma$-consistent, i.e. it can be extended to a $\Gamma$-maxiconsistent set which gives rise to a canonical model. By the truth lemma, this model falsifies $\varphi$. On the other hand, $\Gamma$-closure guarantees that all elements of $\Gamma$ are true in this model. Hence $\varphi$ cannot be entailed by $\Gamma$. $\dashv$

## 7 Translation **L**$\diamond$ $\Rightarrow$ **L**

Versmissen (1996) proves soundness and completeness of the following translation from **L**$\diamond$ to **L**:

Definition 22:

$$
\begin{aligned}
[p] &= p && \text{(p atomic)} && (1)\\
[A \bullet B] &= [A] \bullet [B] && && (2)\\
[A \setminus B] &= [A] \setminus [B] && && (3)\\
[A/B] &= [A]/[B] && && (4)\\
[\diamond_i A] &= t_{i,0} \bullet [A] \bullet t_{i,1} && && (5)\\
[\Box_i^{\downarrow} A] &= t_{i,0} \setminus [A]/t_{i,1} && && (6)\\
[(_i X)_i] &= t_{i,1}, [X], t_{i,1} && && (7)
\end{aligned}
$$

*where $t_{i,0}$ and $t_{i,1}$ are fresh atomic formulas.*

Versmissen's proof is purely syntactic. Completeness of **L**$\diamond$ in horizontal relational interpretation lends itself naturally for a semantic proof, following the strategy of Kurtonina and Moortgat (1997). First we show that every horizontal model for **L**$\diamond$ can be transformed into a model for **L** which verifies the same formulas modulo translation.

Lemma 15: *Let $M = \langle W, <, I, \{R_i | i \in I\}, \{S_i | i \in I\}, V\rangle$ be an arbitrary model for $\boldsymbol{L}\diamond$ and $M'$ be the $\boldsymbol{L}$-model $\langle W, <, V'\rangle$, where $V'$ extends $V$ by mapping $t_{i,0}$ to $R_i$ and $t_{i,1}$ to $S_i$. Then it holds that for all $\boldsymbol{L}\diamond$-formulas and bracketed sequences of $\boldsymbol{L}\diamond$-formulas $X$ that*

$$M, \langle a, b\rangle \models X \text{ iff } M', \langle a, b\rangle \models [X]$$

*Proof:* By induction on the complexity of $X$. The induction base and the induction step for "$\bullet$", "$\setminus$", "/" and sequencing are straightforward.

1. $X = \diamond B, \Rightarrow$ Suppose $M, ab \models \diamond A$. Then there are $c, d$ such that $aR_0c, M, cd \models B$, and $dR_1b$. By induction hypothesis, $M', cd \models [B]$. By the construction of $M'$, $M', ac \models t_0, M', db \models t_1$. Hence $ac \models t_0 \bullet [B]$ and $ab \models t_0 \bullet [B] \bullet t_1 = [\diamond B]$.

2. $\Leftarrow$ Suppose $M', ab \models t_0 \bullet [B] \bullet t_1$. Then there are $c, d$ with $M', ac \models t_0, M', cd \models [B], M', db \models t_1$. By hypothesis, $M, cd \models B$, and by the construction of $M'$, $aR_0c, dR_1b$. Hence $M, ab \models \Diamond B$.

3. $X = \Box^{\downarrow} B, \Rightarrow$ Suppose $M, ab \models \Box^{\downarrow} B$. This entails that $a < b$. Now assume that $M', ab \not\models t_0 \setminus [B]/t_1$. Then there are $c, d$ such that $M', ca \models t_0, M', bd \models t_1, M', cd \not\models [B]$. By hypothesis $M, cd \not\models B$, and by the construction of $M'$, $cR_0a, bR_1d$. By transitivity of $<$, $c < d$, which contradicts the assumption.

4. $\Leftarrow$. Suppose $M', ab \models t_0 \setminus [B]/t_1$, and $M, ab \not\models \Box^{\downarrow} B$. Then there are $c, d$ such that $cR_0a$ (i.e. $M', ca \models t_0$) and $bR_1d$ (i.e. $M', bd \models t_1$). By transitivity, $c < d$, and $M, cd \models B$. By induction hypothesis, $M', cd \models [B]$, which leads to a contradiction.

5. $X = (Y)$ Analogous to $\Diamond$.

$\dashv$

Theorem 4:

$$\vdash_{\mathbf{L}\Diamond} X \Rightarrow A \text{ iff } \vdash_{\mathbf{L}} [X] \Rightarrow [A]$$

Left to right is an easy induction on the length of derivations. For the other direction, assume that $\not\vdash_{\mathbf{L}\Diamond} X \Rightarrow A$. By completeness, there is a model $M$ such that $M \models X, M \not\models A$. By the truth lemma, $M' \models [X], M' \not\models [A]$. By soundness, $\not\vdash_{\mathbf{L}} [X] \Rightarrow [A]$.

$\dashv$

## 8 Conclusion

In this paper we proposed to extend the relational semantics for **L** that was developed in Pankrat'ev (1994) and Andréka and Mikulás (1994) to Moorgat's (1996) multimodal extension **L**$\Diamond$ of **L**. We investigated two such extension, one being inspired by Versmissen's (1996) translation from **L**$\Diamond$ to **L**, and one by the standard Kripke semantics of unary modal operators. We established soundness, weak completeness and strong completeness for both interpretations, thereby following Kurtonina's (1995) strategy of using labeled deduction to construct canonical models. Finally we showed that one of these relational interpretations can be employed to give a semantic proof for the completeness of Versmissen's translation.

These results raise several issues for further research. First, we restricted attention to multimodal Lambek calculi which comprise just one binary mode. It seems worth exploring whether natural relational interpretations are possible for a multimodal system where several binary modes coexist[3] Second, the unary modes that we considered are plain residuation modalities; they neither interact with each other nor with the binary mode. Many applications of multimodal Lambek calculi assume interaction postulates between the modes. It remains to be seen which of these postulates have semantic counterparts under a relational interpretation. Finally, our models where entirely abstract, perhaps it is possible to relate the semantics developed here to more concrete instantiations of relational interpretation that are specific to the intended linguistic applications of the logics under discussion.

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[3] This was justly pointed out by an anonymous referee.

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