Łukasiewicz logic: an introduction

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Kutaisi2011
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Prologue

Betting on Vague Propositions?
It is the eve of the 2006 World Cup Final at Berlin’s Olympic Stadium: Italy is going to play France.
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\[ E = \text{“Italy scores in the match against France”} \]
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Blaise’s Stake

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Ada’s Book

Blaise Pascal (1623 – 1662)

Ada Lovelace (1815 – 1852)
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25 cents now.

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25 cents now.

1€ if Italy scores, 0€ otherwise.

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Probability Theory deals with Events described by Formulae in Classical Logic (=Boolean algebras of Events).
It is the eve of the 2006 World Cup Final at Berlin’s Olympic Stadium: Italy is going to play France.

$$E = \text{“Italy scores late in the match against France”}$$
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Is There a Probability Theory of Events described by such Vague Propositions? (=non-Boolean algebras of Events?)

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Is There a Logic of Vague Propositions?
Is it just Classical Logic?
If not, which Non-Classical Logic is it?

Blaise Pascal (1623 – 1662)
Ada Lovelace (1815 – 1852)
Who Cares, Anyway?

Some Motivating Remarks
Main Motivation

To develop a
Theory of Probabilities of Vague Events
bridging the gap from
Foundations to Applications

Why is it Important?

Because
Vague (or Non-Classical) Events are everywhere and
Classical Probability Theory does not cope well with them
Why is it Important?

Classical Probability Theory does **not** cope well with Vague Events

**Solution 1:**
Insist that vagueness be ruled out.
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- You can precisify a vague event: “Italy will score against France in the last 15’ of the match.”
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Solution 1:
Insist that vagueness be ruled out.

○ You can precisify a vague event: “Italy will score against France in the last 15' of the match.”

○ Then Blaise loses the whole stake if Italy scores at 74’59”, and wins the whole stake if Italy scores at 75’01”.
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Solution 1:
Insist that vagueness be ruled out.

- You can precisify a vague event: “Italy will score against France in the last 15’ of the match.”
- Then Blaise loses the whole stake if Italy scores at 74’59”, and wins the whole stake if Italy scores at 75’01”.
- This violates continuity, our fundamental intuition about the proposition “to score late”: there is no single instant of time that counts as “the first late one”. (Cf. Eubulides’ Sorites Paradox.)
Why is it Important?

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Solution 2:
Use random variables/measure theory directly.
Why is it Important?

Classical Probability Theory does *not* cope well with Vague Events

Solution 2:
Use random variables/measure theory directly.

- If [0,1] is the (normalized) duration of the match, choose a Borel probability measure $\mu$ on it that models "to score late" appropriately.
Why is it Important?

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**Solution 2:**
Use random variables/measure theory directly.

- If $[0,1]$ is the (normalized) duration of the match, choose a Borel probability measure $\mu$ on it that models “to score late” appropriately.
- Then Blaise gets back $\mu([0,t]) \times$ the stake if Italy scores at time $t$. 
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- Problem 2: Even if $\mu$ is given by some Oracle, what is the corresponding measure for, say, "to score early"?
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- Problem 1: What are the admissible/appropriate measures \(\mu\)? **Ad Hoc Models.**
- Problem 2: Even if \(\mu\) is given by some Oracle, what is the corresponding measure for, say, "to score early"? **Algebra of r.v. ≠ Logic of Vague Events.**
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We will look at Łukasiewicz infinite-valued propositional logic. It is the best candidate I know for a logic of vague propositions.
Theories of Vagueness

A Cursory Sketch
The Sorites Paradox

0 From ancient Greek: “The Paradox of the Heap”.
0 Attributed to Eubulides of Miletus, 4th century BC.
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0 1 grain of wheat does not make a heap.
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0 1 grain of wheat does not make a heap.
0 If 1 grain of wheat does not make a heap, then 2 grains of wheat do not.

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0 If \((10^{100}-1)\) grains of wheat do not make a heap, then \(10^{100}\) do not.
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0 ...
0 If (10^{100}-1) grains of wheat do not make a heap, then 10^{100} do not.
0 Hence: 10^{100} grains of wheat do not make a heap.
The Sorites Paradox

0 Chrysippus’ response (inferred from Cicero’s writings):
0 Q. Does 1 grain of wheat make a heap?
0 C. No.
The Sorites Paradox

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  0 Q. *Does 1 grain of wheat make a heap?*
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  0 Q. *Do 2 grains of wheat make a heap?*
  0 C. *No.*
  0 ...
  0 Q. *Do x (some x) grains of wheat make a heap?*
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0 C. \(\text{Silence.}\)

0 ...

You will shortly see that some modern philosophers (the \textit{epistemicists}) arguably side with Chrysippus; many others, however, do not.
The Sorites Paradox

0 Modern response: *Theories of Vagueness.*
0 Initial problem: the monadic predicate $\text{Heap}(x)$ is vague.
0 To explain the paradox away we need a theory of such vague predicates.
0 Any such theory needs some pre-theoretical, or at least theory-neutral, understanding of what a “vague predicate” is.
0 Building on such a common pre-theoretical understanding of vagueness, a plethora of conflicting theories of vagueness has been advanced in the 20th century.
The Sorites Paradox

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0 Any such theory needs some pre-theoretical, or at least theory-neutral, understanding of what a “vague predicate” is.
0 Building on such a common pre-theoretical understanding of vagueness, a plethora of conflicting theories of vagueness has been advanced in the 20th century.
0 So there is no explanation of the Sorites Paradox that is “standard”, in the sense of being most widely accepted.
The monadic predicate \( P(x) := \text{“x is prime”} \), interpreted over the set of natural numbers \( x \geq 1 \), is (absolutely) precise: its extension is the set of prime numbers; its anti-extension is the set of composite numbers; each number either belongs to the extension of \( P \) or to its anti-extension, but not to both; and in principle there is no issue as to whether a given number be prime or composite — though in practice it may be impossible to ascertain which is the case for an astronomic instance of \( x \).
Theory-neutral features of vagueness

Features of a vague predicate.

By contrast, the monadic predicate $R(x) := \text{"x is red"}$, interpreted over the set of all objects, is (to some extent) vague: its extension ought to be the set of all red objects; its anti-extension ought to be the set of all non-red objects; but it may not be clear, even in principle, just which objects do qualify as red, and which as non-red — think of a peculiar tint at the borderline between red and pink.
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The set of all red coats?
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Theory-neutral features of vagueness

Features of a (monadic) vague predicate $R$:

(FV1) $R$ admits *borderline cases* over the intended domain of interpretation $D$, i.e. there are instantiations of $R(x)$ by (a term naming a constant) $c \in D$ such that it is unclear whether $R(c)$ holds or its negation $\neg R(c)$ does.

(FV2) $R$ lacks *sharp boundaries* over the intended domain of interpretation $D$, i.e. there is no clearly defined boundary separating the extension of $R(\cdot)$ from its anti-extension.

(FV3) $R$ is susceptible to a *Sorites series* over the intended domain of interpretation $D$, i.e. there are instantiations of $R(x)$ by $c_1, \ldots, c_n \in D$ such that it is clear that $R(c_1)$ holds, it is clear that $R(c_n)$ does not hold, and it seems at least plausible that if $R(c_i)$ holds then so does $R(c_{i+1})$, for each $i \in \{1, \ldots, n-1\}$. 
Theory-neutral features of vagueness

Features of a vague proposition $p$:

(PV1) $p$ admits borderline cases over the intended set of possible worlds $W$, i.e. there are worlds $w \in W$ such that it is unclear whether $p$ holds in $w$, or its negation $\neg p$ does.

(PV2) $p$ lacks sharp boundaries over the intended set of possible worlds $W$, i.e. there is no clearly defined boundary separating the extension of $p$ — the subset of possible worlds in $W$ at which $p$ is true — from its anti-extension.

(PV3) $p$ is susceptible to a Sorites series over the intended set of possible worlds $W$, i.e. there are possible worlds $w_1, \ldots, w_n \in W$ such that it is clear that $p$ holds in $w_1$, it is clear that $p$ does not hold in $w_n$, and it seems at least plausible that if $p$ holds in $w_i$ then $p$ also holds in $w_{i+1}$, for each $i \in \{1, \ldots, n-1\}$.
Theories of vagueness

Useful Reader: R. Keefe and P. Smith, eds.
Theories of vagueness

Epistemicism: Vagueness as Ignorance
Theories of vagueness

Supervalueationism: Vagueness as Precisifiability
Theories of vagueness

**Contextualism**: Vagueness as dependence from Context
Theories of vagueness

Degree-Based Theories: Vagueness as Truth-in-Degrees
Main Assumption: Truth comes in degrees.

• If $x$ is a clear case of $R$, then $R(x)$ is (fully, classically) true.

• If $x$ is a clear non-case of $R$, then $R(x)$ is (fully, classically) false.

• If $x$ is a borderline case of $R$, then $R(x)$ is true (or false) to a degree.

It may seem natural to say that, in borderline cases, a certain coat is neither clearly red, nor clearly non-red, so that “This coat is red” is neither true nor false. And the further step of then saying that “This coat is red” is true (or false) to some degree may also sound appealing. (Well, does it sound appealing to you?) But we should be aware that taking this direction is a major departure from the roots of logic as we know it, both philosophically and mathematically.
Main Assumption: Truth comes in degrees.

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We are therefore driven into accepting the truth value [Wahrheitswert] of a sentence as constituting its reference [Bedeutung]. By the truth value of a sentence I understand the circumstance that it is true or false. There are no further truth values. For brevity I call the one the True [das Wahre], the other the False [das Falsche].

G. Frege, *On Sense and Reference*, 1892, p. 34.
Frege on Truth

We are therefore driven into accepting the truth value \([Wahrheitswert]\) of a sentence as constituting its reference \([Bedeutung]\). By the truth value of a sentence I understand the circumstance that it is true or false. There are no further truth values. For brevity I call the one the True \([das Wahre]\), the other the False \([das Falsche]\).

G. Frege, *On Sense and Reference*, 1892, p. 34.

In other writings (notably the unpublished *Logik*), Frege makes the following very clear.

- Truth is a primitive notion in logic: it cannot be defined.

- \(\text{True}(p)\) is a peculiar predicate in that it does not admit comparatives: \(p\) is truer than \(q\) is a \textit{façon de parler} lacking genuine logical content.

- (Implicitly.) In particular, \textbf{degrees of truth are non-sense}, according to Fregean orthodoxy.
But *pace* Frege...

- **Many-valued logics** postulate the existence of degrees of truth. (*Caution*: Sometimes motivations are mathematical, not philosophical.)

- Most degree-based theories of vagueness argue that one or other system of many-valued logic is the logic of vague propositions (or predicates).

- By far the majority of such theories make a **far stronger** assumption, namely:

  **Stronger Assumption.** Degrees of truth are modelled by the real unit interval $[0, 1] \subseteq \mathbb{R}$. 
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- **Many-valued logics** postulate the existence of degrees of truth. (*Caution:* Sometimes motivations are mathematical, not philosophical.)

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- By far the majority of such theories make a **far stronger** assumption, namely:

  **Stronger Assumption.** Degrees of truth are modelled by the real unit interval $[0, 1] \subseteq \mathbb{R}$.

**Hint:** With finitely many truth values, it is impossible to deal with borderline cases of borderline cases of ... borderline cases of redness. This is called the *Problem of Higher-Order Vagueness*. 
“Problems worthy of attack prove their worth by hitting back.”

There are forceful objections to the stronger assumption that degrees of truth be identifiable with real numbers. Here are some quotes on the so-called Problem of Artificial Precision.

[Fuzzy logic] imposes artificial precision [. . . While] one is not obliged to require that a predicate either definitely applies or definitely does not apply, one is obliged to require that a predicate definitely applies to such-and-such, rather than to such-and-such other, degree (e.g. that a man 5ft 10in tall belongs to tall to degree 0.6 rather than 0.5).

S. Haack, 1979
“Problems worthy of attack prove their worth by hitting back.”

There are forceful objections to the stronger assumption that degrees of truth be identifiable with real numbers. Here are some quotes on the so called *Problem of Artificial Precision*.

One serious objection to [the many-valued approach] is that it really replaces vagueness with the most incredible and refined precision.

M. Tye, 1989
“Problems worthy of attack prove their worth by hitting back.”

There are forceful objections to the stronger assumption that degrees of truth be identifiable with real numbers. Here are some quotes on the so called
Problem of Artificial Precision.

The degree theorist’s assignments impose precision in a form that is just as unacceptable as a classical true/false assignment. [...] All predications of “is red” will receive a unique, exact value, but it seems inappropriate to associate our vague predicate “red” with any particular exact function from objects to degrees of truth. For a start, what could determine which is the correct function, settling that my coat is red to degree 0.322 rather than 0.321?

R. Keefe, 2000
“Problems worthy of attack prove their worth by hitting back.”

There are forceful objections to the stronger assumption that degrees of truth be identifiable with real numbers. Here are some quotes on the so called Problem of Artificial Precision.

Intuitively, it is not correct to say that there is one unique element of $[0, 1]$ that correctly represents the degree of truth of ‘Bob is bald’, with all other choices being incorrect. [...] we have an affront to intuition [because] [w]e cannot see what could possibly determine that the degree of truth of ‘Bob is bald’ is 0.61 rather than 0.62 or 0.6 [...]

N.J.J. Smith, 2008
“Problems worthy of attack prove their worth by hitting back.”

There are forceful objections to the stronger assumption that degrees of truth be identifiable with real numbers. Here are some quotes on the so called *Problem of Artificial Precision*.

*How could one respond to the problem of artificial precision?*

It turns out that it is not easy to say anything interesting about it, if we do not define formally the logical system involved. There are a lot of different logics that can be based on the assumption that [0,1] is the set of truth values. Whenever one of them claims that such numbers are degrees of truth, you can raise the objection of artificial precision to it.
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However, it is not reasonable to expect that successful responses (if any) to the problem of artificial precision be independent of the details of the underlying logic.
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However, it is not reasonable to expect that successful responses (if any) to the problem of artificial precision be independent of the details of the underlying logic.

Therefore we will now leave theories of vagueness and take a long excursion into formal logic, with the aim of introducing Łukasiewicz logic formally, independently of any intuitive semantics. We will eventually get back to the problem of artificial precision to see whether what we will have learnt can help us with it.