A few selected topics

Non tutto il male vien per nuocere.\textsuperscript{1}

Italian saying.

\textsuperscript{1}Added in Proof. “Not all bad things come to hurt you”. Refers to an embarrassing mistake I made in the preceding talk, concerning the example with the tossing of a coin. In the present talk I took the opportunity to clarify the notions of satisfiability/consistency in Łukasiewicz logic which, as I realised also thanks to my own mistakes in the preceding talk, I had not explained clearly enough.
Clarifications on satisfiability and consistency in $\mathcal{L}$

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<tr>
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Clariﬁcations on satisfiability and consistency in $L$

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Equivalent in classical logic by the Principle of Bivalence.
Clarifications on satisfiability and consistency in $\mathcal{L}$

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Equivalent in classical logic by the Deduction Theorem.
Clarifications on satisfiability and consistency in L

Deduction Theorem for CL

For any \( \alpha, \beta \in \text{FORM} \),

\[
\alpha 
\vdash \beta \quad \text{if, and only if,} \quad \vdash \alpha \rightarrow \beta.
\]
Clarifications on satisfiability and consistency in $L$

**Deduction Theorem for CL**

For any $\alpha, \beta \in \text{FORM}$,

$$\alpha \vdash \beta \text{ if, and only if, } \vdash \alpha \rightarrow \beta.$$ 

The direction $\Rightarrow$ fails in $L$: $\alpha \vdash_L \alpha \odot \alpha$, but $\not\vdash_L \alpha \rightarrow \alpha \odot \alpha$.

(Recall that $\alpha \odot \beta := \neg(\alpha \rightarrow \neg \beta).$)
Clarifications on satisfiability and consistency in $\mathcal{L}$

**Deduction Theorem for CL**

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(Recall that $\alpha \circ \beta := \neg(\alpha \rightarrow \neg \beta)$.)

**Local Deduction Theorem for $\mathcal{L}$**

For any $\alpha, \beta \in \text{FORM}$,

$$\alpha \vdash_\mathcal{L} \beta \text{ if, and only if, } \exists n \geq 1 \text{ such that } \vdash_\mathcal{L} \alpha^n \rightarrow \beta .$$

(Recall that $\alpha^n := \underbrace{\alpha \circ \cdots \circ \alpha}_n$.)
Example. Consider the formula $\alpha(X) = X \land \neg X$.

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Example. Consider the formula $\alpha(X) = X \land \neg X$.

- Is $\alpha$ satisfiable? **No.** There is no evaluation that attributes value 1 to $\alpha$.
- Is $\alpha$ consistent? **No.** $\alpha \vdash L\alpha^2$, but $\vdash L\alpha^2 \leftrightarrow \bot$, so $\alpha \vdash L\bot$.
- Is $\alpha$ unsatisfiable? **Yes.** Whatever you evaluate, the result is $< 1$.
- Is $\alpha$ inconsistent? **Yes.** We saw above that it proves $\bot$, so it proves anything by Ex Falso Quodlibet.
- Is $\alpha$ strongly unsatisfiable? **No.** Evaluate at $1^2$.
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Clarifications on satisfiability and consistency in $L$

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**Which functions are definable?**

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Is Łukasiewicz logic over 1 variable functionally complete w.r.t. functions $[0, 1] \rightarrow [0, 1]$?
Obviously, it cannot be that all functions $[0, 1] \rightarrow [0, 1]$ are definable, e.g. because there are non-continuous functions and we saw that the Łukasiewicz connectives are interpreted by continuous operations.
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To make an educated guess at what the answer is, we need to look at more examples. (At the board.)

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<td>$\neg \alpha$</td>
<td>$w(\neg \alpha) = 1 - w(\alpha)$</td>
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<tr>
<td>$\alpha \lor \beta$</td>
<td>$w(\alpha \lor \beta) = \max {w(\alpha), w(\beta)}$</td>
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<td>$\alpha \land \beta$</td>
<td>$w(\alpha \land \beta) = \min {w(\alpha), w(\beta)}$</td>
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<td>$\alpha \oplus \beta$</td>
<td>$w(\alpha \oplus \beta) = \min {1, w(\alpha) + w(\beta)}$</td>
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Table: Formal semantics of connectives in Łukasiewicz logic.
A function \( f : [0, 1] \rightarrow [0, 1] \) is piecewise linear if it is continuous, and there is a finite set \( \{ L_1, \ldots, L_m \} \) of affine linear functions \( L_i : \mathbb{R} \rightarrow \mathbb{R} \), \( L_i(x) = a_i x + b_i \) for \( a_i, b_i \in \mathbb{R} \), such that, for each \( x \in [0, 1] \), \( f \) agrees with some \( L_i \) (depending on \( x \)). If such a function is such that each \( a_i \) and \( b_i \) can be chosen to be integers, then it is called a \( \mathbb{Z} \)-map.
A function \( f : [0, 1] \rightarrow [0, 1] \) is \textit{piecewise linear} if it is continuous, and there is a finite set \( \{L_1, \ldots, L_m\} \) of affine linear functions \( L_i : \mathbb{R} \rightarrow \mathbb{R} \), \( L_i(x) = a_i x + b_i \) for \( a_i, b_i \in \mathbb{R} \), such that, for each \( x \in [0, 1] \), \( f \) agrees with some \( L_i \) (depending on \( x \)). If such a function is such that each \( a_i \) and \( b_i \) can be chosen to be integers, then it is called a \( \mathbb{Z} \)-map.

\[ \text{A piecewise linear function \([0, 1] \rightarrow \mathbb{R}\)} \]
McNaughton’s Theorem in 1 variable

A function $f : [0, 1] \rightarrow [0, 1]$ is definable by a formula in Łukasiewicz logic if, and only if, it is a $\mathbb{Z}$-map.
McNaughton’s Theorem in 1 variable

A function $f : [0, 1] \to [0, 1]$ is definable by a formula in Łukasiewicz logic if, and only if, it is a $\mathbb{Z}$-map.

By appropriate generalisation of the notion of $\mathbb{Z}$-maps to functions $[0, 1]^\kappa \to [0, 1]$, the theorem extends to arbitrary sets of variables.
The Farey tree.
The Farey formulæ (related to Schauder bases).

Recall: \( \alpha \ominus \beta = \neg(\alpha \rightarrow \beta) \); truncated subtraction.
Cauchy’s Thm. Every rational number in $(0,1)$ occurs, automatically in reduced form, as the mediant of the numbers in some node of the Farey tree exactly once. (*Added in proof.* The mediant of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$.)
**Thm.** Given any maximal consistent theory $\Theta$ in $\mathcal{L}$ (over 1 variable $X$) that is not just $\{X\}^{\mathcal{L}}$ or $\{-X\}^{\mathcal{L}}$, there is exactly one node in the tree such that there is an integer $n \geq 1$ with $\Theta = \{n(L \land R)\}^{\mathcal{L}}$. Moreover, this $n$ is unique and equals the denominator of the mediant of the node.
Where to learn more

A basic but comprehensive introduction to Łukasiewicz logic and its algebraic counterpart, MV-algebras:

A much more advanced treatment of the mathematics of MV-algebras:

 Łukasiewicz logic is part of the considerably larger hierarchy of Petr Hájek’s mathematical fuzzy logics:

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Epilogue: Betting on vague propositions, again

Pierre Fermat (1601 – 1665)
Blaise Pascal (1623 – 1662)

Historiographic cliché: Probability theory begins in 1654, with the correspondence between Fermat and Pascal on the problem of points, proposed to them by the Chevalier de Méré (born Antoine Gombaud).
ANNÉE 1654.

LXIX.

FERMAT À PASCAL (1).

1654.

(Oeuvres de Pascal, 1779, IV, p. 411–412.)

Monsieur,

Si j'entreprends de faire un point avec un seul dé en huit coups; si nous convenons, après que l'argent est dans le jeu, que je ne jouerai pas le premier coup, il faut, par mon principe, que je tire du jeu du total pour être désintéressé, à raison dudit premier coup.

Que si encore nous convenons après cela que je ne jouerai pas le second coup, je dois, pour mon indemnité, tirer le 6ème du restant, qui

Part of Pascal's reply:

2. Votre méthode est très-sûre et est celle qui m'est la première venue à la pensée dans cette recherche; mais, parce que la peine des combinaisons est excessive, j'en ai trouvé un abrégé et proprement une autre méthode bien plus courte et plus nette, que je voudrois vous pouvoir dire ici en peu de mots: car je voudrois désormais vous ouvrir mon cœur, s'il se pouvoit, tant j'ai de joie de voir notre rencontre. Je vois bien que la vérité est la même à Toulouse et à Paris.
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Neither Pascal nor Fermat explicitly bring logic to bear on probability. What does logic have to do with the theory of probability?
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$$1, 1, 3, 6, 2, 4, 6, 4.$$
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The set $S$ of all possible outcomes is called the sample space.
One key primitive notion in probability theory is that of event. An important tradition in the subject regards probability theory as an attempt to model the possible outcomes of idealised experiments. Thus, in the letter from Fermat to Pascal quoted above, the experiment is a sequence of eight throws of a die (with faces numbered from 1 to 6). The possible outcomes of the experiment are all possible sequences of points in the eight throws; one such, e.g., is

$$1, 1, 3, 6, 2, 4, 6, 4.$$  

The set $S$ of all possible outcomes is called the sample space. Certain subsets of $S$ (not necessarily all) are then selected as having special interest for the problem at hand; they form the collection $\mathcal{E}$ of events.
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Thus, returning to Fermat’s example, the set consisting of the two sequence of points

\[1, 1, 1, 1, 1, 1, 1, 1\] and \[6, 6, 6, 6, 6, 6, 6, 6\]

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\[ 1, 1, 1, 1, 1, 1, 1, 1 \quad \text{and} \quad 6, 6, 6, 6, 6, 6, 6, 6 \]

corresponds to an event, because it may be described by a proposition. Say,

“Either one observes, as the outcome of the experiment, the smallest possible point at each throw, or else one observes the largest possible point at each throw.”
Boole on events vs. propositions:

6. Before we proceed to estimate to what extent known methods may be applied to the solution of problems such as the above, it will be advantageous to notice, that there is another form under which all questions in the theory of probabilities may be viewed; and this form consists in substituting for events the propositions which assert that those events have occurred, or will occur; and viewing the element of numerical probability as having reference to the truth of those propositions, not to the occurrence of the events concerning which they make assertion.
Keynes on events vs. propositions:

CH. I

FUNDAMENTAL IDEAS

4. With the term "event," which has taken hitherto so important a place in the phraseology of the subject, I shall dispense altogether.\footnote{1} Writers on Probability have generally dealt with what they term the "happening" of "events." In the problems which they first studied this did not involve much departure from common usage. But these expressions are now used in a way which is vague and ambiguous; and it will be more than a verbal improvement to discuss the truth and the probability of \textit{propositions} instead of the occurrence and the probability of \textit{events}.\footnote{2}

John Maynard Keynes, \textit{A Treatise on Probability}, p. 5, Cambridge 1920
This shift of perspective is more than a verbal improvement in that
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To summarise:

*The rôle of logic in the theory of probability is to provide a formal model for the notion of event.*
Let us now turn to probabilities.
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Fix a theory $\Theta$. A probability assignment (relative to $\Theta$) is a function

$$P : \text{FORM} \rightarrow [0, 1]$$

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1. **(K0)** $P(\alpha) = P(\beta)$ whenever $\Theta \vdash \alpha \leftrightarrow \beta$.
2. **(K1)** $P(\bot) = 0$ and $P(\top) = 1$.
3. **(K2)** $P(\alpha \lor \beta) = P(\alpha) + P(\beta)$ whenever $\Theta \vdash (\alpha \land \beta) \leftrightarrow \bot$. 
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How do we know that these axioms capture our intuitions about probability (if any)?
The Ramsey-de Finetti **Dutch book argument** (1926, 1937), along with its later utility-based version by L. Savage (1954), provides one possible answer.
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Frank P. Ramsey (1903 – 1930)

Bruno de Finetti (1906 - 1985)
Consider a finite family of events $\mathcal{E} = \{E_1, \ldots, E_n\}$, and a function $f: \mathcal{E} \to [0, 1]$. When is $f$ to be considered an assignment of probabilities?
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True to her palindromic name, though, Ada also accepts reverse bets. That is, she also accepts Blaise’s negative stakes $\sigma_i < 0$, to the effect that she must hand $|\sigma_i|\beta(E_i)$ euros to Blaise, with the agreement that $|\sigma_i|\omega(E_i)$ euros shall be paid back by Blaise to Ada in the possible world $\omega$. 
True to her palindromic name, though, Ada also accepts reverse bets. That is, she also accepts Blaise’s negative stakes $\sigma_i < 0$, to the effect that she must hand $|\sigma_i|\beta(E_i)$ euros to Blaise, with the agreement that $|\sigma_i|\nu(E_i)$ euros shall be paid back by Blaise to Ada in the possible world $w$.

Hence, the final balance of Ada’s book $\beta : \mathcal{E} \rightarrow [0, 1]$ is given by

$$\sum_{i=1}^{n} (\sigma_i \beta(E_i) - \sigma_i \nu(E_i)),$$

where it is understood that money transfers are oriented so that ‘positive’ means ‘Blaise-to-Ada’.
Now de Finetti’s Criterion states that Ada’s book should be rejected (i.e. it is not a rational assignment) if and only if it is incoherent or Dutch, meaning that Blaise can choose stakes so as to make Ada suffer a sure loss.
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An assignment of numbers from $[0, 1]$ to the events in $E$ is coherent if it is not incoherent.
Now we use the Boole-Keynes idea, and we regard the events in $\mathcal{E}$ simply as a family of formulæ in classical logic. We can assume without loss of generality that $\mathcal{E}$ is closed under deduction, i.e. is a theory. Now it makes sense to ask whether an assignment of numbers in $[0, 1]$ to $\mathcal{E}$ satisfies Kolmogorov’s axioms in the form reviewed above.
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Coherent assignments to $\mathcal{E}$ are the same thing as assignments that satisfy Kolmogorov’s axioms.
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The theorem provides a fundamental operational explanation of Kolmogorov’s axioms for finitely additive probabilities.
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Fix a theory $\Theta$ in Ł. A **probability assignment** or **state** (relative to $\Theta$) is a function

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Consider a finite family of many-valued events $\mathcal{E} = \{E_1, \ldots, E_n\}$, i.e. propositions in $\mathcal{L}$, and a function $f : \mathcal{E} \rightarrow [0, 1]$. Can we give an operational characterisation of the circumstance that $f$ is a state?
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Ada is still palindromic, and Blaise can ask her to swap rôles by placing a bet with negative stake.
Epilogue

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**De Finetti No-Dutch-Book Theorem for Ł**

Coherent assignments to $\mathcal{E}$ are the same thing as assignments that satisfy the axioms for states.

The theorem provides a fundamental operational explanation of the axiom for states. It is the beginning of the (nascent) theory of probability of events described by formulæ in a non-classical logic.
Thank you for your attention.