How to Cover without Lifting Relations

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TbiLLC 2011
# The Classical Cover Modality

## Standard Syntax of Modal Logic

\[ L_{\Box, \Diamond} \ni \phi, \psi ::= p \mid \phi \land \psi \mid \phi \lor \psi \mid \neg \phi \mid \Box \phi \mid \Diamond \phi \quad (p \in V) \]

## Modal Logic in Terms of the Cover Modality

\[ L_{\nabla} \ni \phi, \psi ::= p \mid \phi \land \psi \mid \phi \lor \psi \mid \neg \phi \mid \nabla \Phi \quad (p \in V, \Phi \subseteq_f L_{\nabla}) \]

## Semantics

Suppose \( M = (W, \sigma : W \to \mathcal{P}(W), \pi : W \to \mathcal{P}(V)) \) is a Kripke model.

\[ x \models \nabla \Phi \text{ iff } \begin{align*}
& \forall \phi \in \Phi \exists y \in \sigma(x). y \models \phi \\
& \forall y \in \sigma(x) \exists \phi \in \Phi. y \models \phi
\end{align*} \]

"\( \Phi \) and the successors of \( x \) mutually cover one another"
Why the Cover Modality?

Back and Forth Translation

Forth.

\[ \nabla \Phi \equiv \Box \bigvee_{\phi \in \Phi} \phi \land \bigvee_{\phi \in \Phi} \Diamond \phi \]

Back.

\[ \Box \phi \equiv \nabla \{ \phi \} \lor \nabla \emptyset \]
\[ \Diamond \phi \equiv \nabla \{ \phi, T \} \]

(we don’t lose any expressiveness)

Correspondence between Syntax and Semantics

- Kripke models \((W, \sigma, \pi)\) come with a \textit{structure} \(\sigma : W \to \mathcal{P}(W)\)
- \(\nabla\)-formulas come with a \textit{constructor} \(\mathcal{P}_f(\mathcal{L}_\nabla) \to \mathcal{L}_\nabla\)

\((\textit{finite} \text{ powersets give \textit{finitary} languages})\)

General Recipe?

\[
\begin{align*}
\text{Semantics: Structures} & \quad \sigma : W \to TW \\
\text{Syntax: Constructors} & \quad \nabla : T_f \mathcal{L} \to \mathcal{L}
\end{align*}
\]
Experiment: Probabilistic Frames

Discrete Probability Distributions

\[ DX = \{ \mu : X \rightarrow [0, 1] \mid \sum_{x \in X} \mu(x) = 1, \#\{x \in X \mid \mu(x) \neq 0\} < \infty \} \]

Probabilistic Kripke Models

\[ M = (W, \sigma : W \rightarrow D(W), \pi : W \rightarrow P(V)) \]

(these are discrete Markov chains)

Probabilistic Modal Logic

\[ L_\nabla \ni \phi, \psi ::= p \mid \phi \land \psi \mid \phi \lor \psi \mid \nabla \Phi \]

(\( p \in V, \Phi \in D(L_\nabla) \))

(probability distributions over formulae are formulae)
Satisfaction for Probabilistic Modal Logic

Semantics and the Magic Square

Ssee $\mathcal{M} = (W, \sigma, \pi)$ is a probabilistic model and $\nabla \mu \in \mathcal{L}_{\nabla}$, i.e. $\mu \in \mathcal{D}(\mathcal{L}_{\nabla})$.

Then $w \models \nabla \mu$ iff we can fill the ‘magic square’

<table>
<thead>
<tr>
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<th>$w_1$</th>
<th>$w_2$</th>
<th>$\cdots$</th>
<th>$w_k$</th>
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<tbody>
<tr>
<td>$\phi_1$</td>
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<td>$q_1$</td>
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<td>$\vdots$</td>
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<td>$\phi_n$</td>
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<td>$\sum$</td>
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<td>$p_2$</td>
<td>$\cdots$</td>
<td>$p_k$</td>
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</tr>
</tbody>
</table>

- $p_j = \sigma(w)(x_j)$ is prob of $x_j$
- $q_i = \mu(\phi_i)$ is prob of $\phi_i$
- $w/\phi$-entry is 0 if $x \not\models \phi$

can be filled according to the rules on the right.

Question.

How far does this generalisation carry? Can we automagically construct magic squares?
White Covers: The General Principle

Definition (\(T\)-models)

Suppose \(T : \text{Set} \to \text{Set}\) is a functor. Then \(T\)-models are triples \((W, \sigma, \pi)\) with \(\sigma : W \to TW\) and \(\pi : W \to \mathcal{P}(W)\).

Definition (\(T\)-Language)

Write \(T_f(X) = \bigcup \{TY \mid Y \subseteq_f X\}\) for the finitary part of \(T\).

\[\mathcal{L}_\nabla^T \ni \phi, \psi ::= p \mid \phi \land \psi \mid \phi \lor \psi \mid \neg \phi \mid \nabla \Phi \quad (p \in V, \Phi \in T_f \mathcal{L}_\nabla)\]

Definition (Semantics)

If \(R \subseteq X \times Y\) is a relation, write

\[\hat{T}(R) = \{(s, t) \in TX \times TY \mid \exists w \in TR. T\pi_1(w) = s \text{ and } T\pi_2(w) = t\}\]

for the relation lifting of \(T\) and put

\[w \models \nabla \Phi \iff (\sigma(w), \Phi) \in \hat{T}(\models)\]
Results and Limitations

Some Results (*terms and conditions apply*)

- Bisimulation invariance and Hennesy-Milner Property [Moss]
- Complete Axiomatisation(s) [Kupke, Kurz, Venema]
- Fixpoint Logics / Distributive Law [Venema]

Limitations: Compatibility with Relational Composition

Required: \( \hat{T}(R \circ S) = \hat{T}R \circ \hat{T}S \)
where \( \hat{T} \) is the relation lifting of \( T \) (even for bisimulation invariance).

Examples (*t’s & c’s fail*)

- Neighbourhood frames: \( W \to \mathcal{P}\mathcal{P}(W) \)
- Monotone nbhd frames: \( W \to \{ N \in \mathcal{P}\mathcal{P}(W) : N \ \text{upclosed} \} \)
- Selection function frames: \( W \to (\mathcal{P}(W) \to \mathcal{P}(W)) \)
Our Approach

White Nablas: $\mathcal{L}_\nabla$

Syntax.
\[ \nabla : T_f \mathcal{L}_\nabla^T \rightarrow \mathcal{L}_\nabla \]

Semantics.
\[ x \models \nabla \Phi \iff \sigma(x) \hat{T}(\models) \Phi \]

Black Nablas: $\mathcal{L}_\blacktriangle$

Syntax.
\[ \blacktriangle : T_C(\Sigma) \rightarrow \mathcal{L}_\blacktriangle^T \]

Semantics.
\[ x \models \blacktriangle \Phi \iff T(t) \circ \sigma(x) = \Phi \]

(\(\Sigma \subseteq f \mathcal{L}_\blacktriangle^T\), \(C(\Sigma)\) are \(\neg\)-complete subsets and \(t\) is the local theory map.)

Conceptual Digression

- satisfaction for \(\mathcal{L}_\nabla\) involves \(\hat{T}(\models)\) which can fail
- satisfaction for \(\mathcal{L}_\blacktriangle\) involves \(T(t)\) which always works
**Example 1: ▼ for Kripke Frames**

\[ \Sigma = \{ p, q, r \} \]

\[ \Phi = \{ \{ p, q, \neg r \} \} \]

\[ w \models \nabla \Phi \]
Example 2: ▼ for Probabilistic Frames

\[ \Sigma = \{p, q\} \]

\[ p, q, \neg r \]

\[ \Phi = 0.2 \cdot \{p, q\} + 0.2 \cdot \{p, q\} + 0.6 \cdot \{\neg p, \neg q\} \]

\[ w \models \nabla 0.2 \cdot \{p, q\} + 0.2 \cdot \{p, q\} + 0.6 \cdot \{\neg p, \neg q\} \]

\[ = \nabla 0.4 \cdot \{p, q\} + 0.6 \cdot \{\neg p, \neg q\} \]
Black Nablas, Formally

**Definition (Syntax)**

Suppose $T : \text{Set} \to \text{Set}$.

$$\mathcal{L}_{\nabla}^T \ni \phi, \psi ::= p \mid \phi \land \psi \mid \phi \lor \psi \mid \neg \phi \mid \nabla \Phi$$

where $p \in V$ and $\Phi \in TC(\Sigma)$ for $\Sigma \subseteq \mathcal{L}_{\nabla}^T \setminus \neg \mathcal{L}_{\nabla}^T$.

(recall $C(\Sigma) = \{ \Delta \subseteq \Sigma \cup \neg \Sigma \mid \forall \phi \in \Sigma. (\phi \in \Delta \text{ or } \neg \phi \in \Delta) \}$)

**Definition (Semantics)**

Given a $T$-model $\mathbb{M} = (W, \sigma : W \to TW, \pi : W \to \mathcal{P}(V))$, put

$$w \models \nabla \Phi \iff T(t \upharpoonright \Sigma) \circ \sigma(w) = \Phi$$

where $t \upharpoonright \Sigma : w \mapsto \{ \phi \in \Sigma \cup \neg \Sigma \mid \mathbb{M}, w \models \phi \}$ is the ($\Sigma$-) local theory map.
Examples and Questions

Examples
- Kripke frames
- neighbourhood frames
- monotone nbhd frames
- probabilistic frames
- conditional frames
- etc.

What is the benefit of ▼?
- Bisimulation invariance and the Hennessy-Milner property
- The finite model property
- Conjunction and Negation Elimination
- Simple Tableaux Calculus

in a uniform framework not requiring that relation lifting is well-behaved.
Translatability for Kripke Semantics

Kripke Semantics: $\mathcal{L}_{\Box,\Diamond}$ to $\mathcal{L}^P_{\Diamond}$

$\Box \phi = \Diamond \emptyset \lor \Diamond \{\{\phi\}\}$

$\Diamond \phi = \Diamond \{\{\phi\}\} \lor \Diamond \{\{\phi\}\}, \{\neg \phi\}\}$

(for us, this direction would be enough)

Kripke Semantics: $\mathcal{L}^T_{\Diamond}$ to $\mathcal{L}_{\Box,\Diamond}$

$c \models \Diamond \Phi \iff c \models \Box \bigvee_{\alpha \in \Phi} (\land \alpha \land \neg \bigvee (\Sigma \setminus \alpha)) \land \bigwedge_{\alpha \in \Phi} \Diamond (\land \alpha) \land \neg \bigvee (\Sigma \setminus \alpha)$
Translatability for Monotone Neighbourhood Frames

Monotone Neighbourhood Models

\[ M = (W, \sigma : W \rightarrow \mathcal{N}(W), \pi : W \rightarrow \mathcal{P}(V)) \]

(recall \( \mathcal{N}(W) = \{ N \in \mathcal{PP}(X) | N \text{ upclosed}\} \))

Semantics

\[ M, w \models \square \phi \iff \{ w' | w' \models \phi \} \in \sigma(w) \]

('\( \phi \) is a neighbourhood of \( w \'))

Translation: \( \mathcal{L}_{\square, \Diamond} \) to \( \mathcal{L}_{\nabla}^\mathcal{P} \)

\[ \square \phi \equiv \nabla \uparrow \{ \alpha_0 \} \lor \nabla \uparrow \{ \alpha_0, \alpha_1 \} \]

where \( \alpha_0 = \{ \{ \phi \} \} \), \( \alpha_1 = \{ \neg \phi \} \).
Bisimulation Invariance

**Definition (T-morphisms)**

A *T-morphism* \( f : (W, \sigma, \pi) \to (W', \sigma', \pi') \) is a map \( f : W \to W' \) such that

\[
\begin{align*}
W & \xrightarrow{f} W' \\
\sigma & \downarrow \quad \downarrow \sigma' \\
TW & \xrightarrow{Tf} TW'
\end{align*}
\]

and

\[
\begin{align*}
W & \xrightarrow{f} W' \\
\pi & \downarrow \quad \downarrow \pi' \\
\mathcal{P}(V) & \xrightarrow{f} \mathcal{P}(V)
\end{align*}
\]

commute. A pair \((w, w') \in W \times W'\) is *behaviourally equivalent* if it can be identified by a pair of *T*-morphisms.

**Special Case: Kripke Frames, i.e. \( T = \mathcal{P} \)**

- \( \mathcal{P}\)-morphisms are \( p \)-morphisms aka functional bisimulations
- behavioural equivalence is bisimilarity
Proposition (Morphisms preserve Semantics)

Let $f : (W, \sigma, \pi) \to (W', \sigma', \pi')$ be a $T$-morphism. Then, for all $\phi \in \mathcal{L}_\Box^T$:

$$w \models \phi \iff f(w) \models \phi.$$ 

Proof. For $\phi = \Box \Psi$: $w$ and $f(w)$ inductively have the same local theories.

Corollary (Behavioural Equivalence implies Logical Equivalence)

Suppose that $(w, w')$ are behaviourally equivalent. Then, for all $\phi \in \mathcal{L}_\Box^T$:

$$w \models \phi \iff w' \models \phi.$$ 

Proof. $w \models \phi \iff f(w) \models \phi \iff g(w') \models \phi \iff w' \models \phi$. 
From Logical to Behavioural Equivalence

**Proposition**

If $T$ is finitary ($\simeq$ finitely branching) and $\sim$ is logical equivalence, then

$$(W, \sigma, \pi) \rightarrow (W/\sim, \sigma/\sim, \pi/\sim)$$

is a well-defined $T$-morphism for all $T$-models $(W, \sigma, \pi)$.

*Proof.* Find $W_0 \subseteq f W$ with $\sigma(w), \sigma(w') \in TW_0$ and observe that $\sim\upharpoonright_{W_0 \times W_0}$ can be characterised by finitely many formulae.

**Corollary**

If $T$ is finitary, then logical and behavioural equivalence coincide.

*Proof.* $w \sim w' \iff [w']_\sim = [w]_\sim$. 

Pattinson and Santocanale (ICL and UProv) How to Cover without Lifting Relations

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Coherent Models and the Truth Lemma

**Definition (Coherent Models)**

Let $\Delta \subseteq L^T_\Box$ be negation closed. Then $M = (W, \sigma, \pi)$ is coherent over $\Delta$ if

- $W = \{ \Theta \subseteq \Delta \mid \Theta$ maximally satisfiable $\}$
- $\Box \Phi \in \Theta \iff T(t \uparrow \Phi) \circ \sigma(\Theta) = \Phi$
- $p \in \Theta \iff p \in \pi(\Theta)$

**Lemma (Truth Lemma)**

If $M = (W, \sigma, \pi)$ is coherent over $\Delta$, then

$$\Theta \models \phi \iff \phi \in \Theta$$

for all $\phi \in \Delta$.

Existence Lemma and the Small Model Property

**Lemma (Existence Lemma)**

*If $\Delta$ is finite and negation closed, then coherent models exist.*

**Proof.** For every maximally satisfiable subset $\Theta \subseteq \Delta$ pick $w \models \Theta$ and define $\sigma(\Theta)$ by replacing points with local theories.

**Corollary (Small Model Property)**

*If $\phi \in \mathcal{L}_T^\downarrow$ is satisfiable, then $\phi$ is satisfiable in an exponential-size model.*

**Proof.** Choose $\Delta$ to consist of the subformulas of $\phi$ and their negations.
Conjunction Elimination

Definition (Conjunction under $T$)

Let $\Phi_i \in TC(\Sigma_i)$ for $i = 1, 2$. Then

$$\Phi_1 \land \Phi_2 = \{ \Delta \in TC(\Sigma_1 \cup \Sigma_2) \mid T(\lambda\Theta.\Theta \cap (\Sigma_i \cup \neg \Sigma_i)) = \Phi_i, i = 1, 2 \}$$

denote the 'conjunction under $T$' of $\Phi_1$ and $\Phi_2$.

Note. Conjunction under $T$ gives all consistent possibilities to satisfy both $\Box \Phi_1$ and $\Box \Phi_2$.

Lemma (Conjunction Elimination Lemma)

$$w \models \Box \Phi_1 \land \Box \Phi_2 \iff \exists \Psi \in \Phi_1 \land \Phi_2. w \models \Box \Psi$$

Proof. If $\Box \Phi_1 \land \Box \Phi_2$ is satisfiable, put $\Psi = T(t \mid_{\Sigma_1 \cup \Sigma_2} \circ \sigma(w))$. 
Negation Elimination

**Definition (Negation under \( T \))**

Let \( \Phi \in TP(\Sigma) \). Then

\[
\neg \Phi = \{ \Delta \in TC(\Sigma) \mid \Delta \neq \Phi \}
\]

denotes the *negation under \( T \)* of \( \Phi \).

**Note.** Negation under \( T \) gives *all possibilities* to satisfy \( \neg \Box \Phi \).

**Lemma (Negation Elimination)**

\[
w \models \neg \Box \Phi \iff \exists \Psi \in \neg \Phi. w \models \Psi
\]

**Proof.** Put \( \Psi = T(t \upharpoonright \Sigma) \circ \sigma(w) \).
Tableaux Calculus

Extra Assumption
Suppose that $TX$ is finite whenever $X$ is finite.

Prolegomena

*Sequents.* finite subsets $\Gamma, \Delta, \ldots$ of $L^T$, read conjunctively.

*Tableaux.* Sequent labelled trees constructed according to rules (below)

*Closed Tableaux.* Maximal such trees.

*Completeness.* Unsatisfiability of $\Gamma \iff$ existence of closed tableau with root $\Gamma$.

Tableau Rules: Propositional Part

\[
\begin{align*}
(\land) & \quad \frac{\Gamma, \phi \land \psi}{\Gamma, \phi, \psi} \\
(\lor) & \quad \frac{\Gamma, \phi \lor \psi}{\Gamma, \phi} \quad \frac{\Gamma, \phi \lor \psi}{\Gamma, \psi} \\
(Ax) & \quad \frac{\Gamma, p, \neg p}{\Gamma}
\end{align*}
\]
Modal Rules

Tableau Rules: Propositional Rules plus

\[
\begin{align*}
&\quad \frac{\neg \Box \Phi, \Gamma}{\Box \Psi, \Gamma \mid \Psi \in \neg \Phi} \quad \frac{\Box \Phi_1 \land \Box \Phi_2, \Gamma}{\Box \Psi, \Gamma \mid \Psi \in \Phi_1 \land \Phi_2} \\
&\quad (\text{Ing}) \frac{\Box \Phi, \Gamma}{\Psi} (\Gamma \subseteq V \cup \neg V \text{ consistent}, \Psi \in \text{Ing}(\Phi))
\end{align*}
\]

where \( \text{Ing}(\Phi) = \bigcap \{ \Psi \subseteq C(\Sigma) \mid \Phi \in T\Psi \} \) are the ingredients of \( \Phi \).

Lemma (Invertibility)

\textit{The premise of a rule is satisfiable iff one if its conclusions is satisfiable.}

\textit{Proof.} For (Ing), choose satisfying models for all conclusions and glue.

Theorem (Completeness)

\textit{The tableau calculus for } \mathcal{L}^T_{\Box} \text{ is complete.}

\textit{Proof.} By invertibility it suffices to observe that all tableaux are finite.
Conclusions

Conceptual Achievements

- *uniform* framework for designing logics over large class of models
- ‘standard’ languages are encodable
- removed relation-lifting barrier

Technical Achievements

- bisimulation invariance and Hennessy-Milber property
- small model property
- complete axiomatisation

Loose Ends

- fixpoint extensions (← non-monotonicity?)
- extensions: nominals, global modality . . .
- satisfiability games and automata