Rules of Inference
Lecture 3
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Today

- Unification theory
- Proof systems
- Open problems
Unification theory
Unification theory

Given two terms $s$ and $t$, is there a substitution $\sigma$ such that $\sigma s \equiv \sigma t$?

**Applications:** computer science, linguistics, . . .

In a logic $L$: $A \equiv B$ is $\vdash_L A \leftrightarrow B$.

The study of substitutions $\sigma$ such that $\vdash_L \sigma A$.

**Dfn** $\sigma$ is a **unifier** of $A$ iff $\vdash \sigma A$.

**Dfn** $\tau \leq \sigma$ iff for some $\tau'$ for all atoms $p$: $\vdash \tau(p) \leftrightarrow \tau'\sigma(p)$.

**Dfn** $\sigma$ is a **maximal** unifier (mu) of $A$ if among the unifiers of $A$ it is maximal.

**Dfn** A unifier $\sigma$ of $A$ is a **mgu** if $\tau \leq \sigma$ for all unifiers $\tau$ of $A$.

**Dfn** A set of unifiers is **complete** for $A$ if all unifiers of $A$ are less general ($\leq$) than a unifier in the set.

**Note** Projective unifiers are most general unifiers.
Unification types

**Ex** $p$ has a mgu in any intermediate logic: $\sigma(p) = \top$. In intermediate logics, every consistent formula has a unifier.

**Dfn** A logic has unification type

*unitary* if every unifiable formula has a mgu;

*finitary* if it is not unitary and every unifiable formula has a finite complete set of mus;

*infinitary* if it is not finitary and every unifiable formula has a finite or infinite complete set of mus;

*nullary* if none of the above.
Valuations and substitutions

**Thm** If $\nu_I(A) = 1$, then $\sigma_I^A$ is a mgu of $A$ in CPC.

**Cor** CPC has unitary unification.
**No unitary unification**

*Note* IPC does not have unitary unification.

*Prf* $A = p \lor \neg p$ has no mgu. For consider $\sigma_0$ and $\sigma_1$ where

$$
\begin{align*}
\sigma_0(p) &= \top & \sigma_0(q) &= q & \sigma_1(p) &= p & \sigma_1(q) &= \bot.
\end{align*}
$$

Neither $\sigma_0 \leq \sigma_1$ nor $\sigma_1 \leq \sigma_0$. So $A$ has no mgu.

If $\tau$ is unifier of $A$, then because of the disjunction property $\vdash_{\text{IPC}} \tau p$ or $\vdash_{\text{IPC}} \neg \tau p$. Hence $\tau \leq \sigma_0$ or $\tau \leq \sigma_1$. Hence $A$ has a finite set of mus: $\{\sigma_0, \sigma_1\}$.

*Note* Many modal logics do not have unitary unification.
Method of proof

**Thm** If there is a set of admissible rules $\mathcal{R}$ such that for every formula $A$ there is a finite set of projective formulas $\Pi_A$ such that

$$\bigvee \Pi_A \vdash^L A \vdash^R \Pi_A,$$

then $\mathcal{R}$ is a basis for the admissible rules of $L$ and $L$ has finitary or unitary unification.

**Prf**
Let for $B \in \Pi_A$, $\sigma_B$ be its projective unifier and let $C$ be the set of these unifiers. If $\tau$ is a unifier of $A$, then it has to be a unifier of $B$ for some $B \in \Pi_A$. Therefore $\tau \leq \sigma_B$, proving that $C$ is complete. \qed
**Thm (Ghilardi)**
The unification type of IPC, K4, S4, GL is finitary. In KC and S4.3 it is unitary.

**Thm (Dzik ’06)**
All intermediate logics with unitary unification are extensions of KC. All intermediate logics with finitary unification are extensions of the logic of the fork. Similar for S4.3 in modal logic.

**Thm (Marra & Spada ’11)**
Łukasiewicz logic has nullary unification type. Formulas do not have finite projective approximations.
Proof systems
Proof systems

In many intermediate and modal logics admissibility is decidable. Many intermediate and modal logics have a decent basis for their admissible rules. Can admissibility in these logics be captured by a decent proof system? Yes: a sequent calculus to reason about rules consisting of sequents.
**Dfn** A *generalized sequent rule (gs-rule)* is an expression

\[ \mathcal{G} \triangleright \mathcal{H}, \]

where \( \mathcal{G} \) and \( \mathcal{H} \) are sets of sequents.

\( \mathcal{G} \triangleright \mathcal{H} \) is *admissible* if \( \mathcal{G} \vdash \mathcal{H} \), which is short for

\[ \{ I(S) \mid S \in \mathcal{G} \} \vdash \{ I(S) \mid S \in \mathcal{H} \}. \]

**Aim:**
A proof system \( \text{GAL} \) for gs-rules such that \( \vdash_{\text{GAL}} \mathcal{G} \triangleright \mathcal{H} \) iff \( \mathcal{G} \vdash_{\text{L}} \mathcal{H} \).
Proof systems

**Dfn** For a modal logic L, GAL consists of (G3 a calculus for CPC):

**Right Logical Rules**

\[
\frac{G 	riangleright S_1, \mathcal{H}}{G 	riangleright S, \mathcal{H}} \quad \frac{G 	riangleright S_2, \mathcal{H}}{G 	riangleright S, \mathcal{H}} \quad \text{for every rule } \frac{S_1}{S} \frac{S_2}{S} \text{ of } \text{G3}
\]

**Left Logical Rules**

\[
\frac{G, S_1, S_2 	riangleright \mathcal{H}}{G, S 	riangleright \mathcal{H}} \quad \text{for every rule } \frac{S_1}{S} \frac{S_2}{S} \text{ of } \text{G3}
\]

**Visser Rules**

\[
\frac{G, S, S_1 	riangleright \mathcal{H}}{G, S 	riangleright \mathcal{H}} \quad \ldots \quad \frac{G, S, S_n 	riangleright \mathcal{H}}{G, S 	riangleright \mathcal{H}} \quad \text{if } \frac{S}{\{S_1, \ldots, S_n\}} \text{ is a rule in the basis}
\]

and some other rules ...
Proof systems

Thm GAK4, GAS4 and GAGL are sound and complete proof systems for admissibility in K4, S4 and GL.

Cor Admissibility in K4, S4 and GL is decidable.
Open problems: to (wish to) do
Ways

Four approaches:
  - characteristic
  - categorical
  - canonical
  - syntactic
**Predicate logic**

*Dfn* The language $\mathcal{L}$ consists of predicate and function symbols, variables, the connectives $\land, \lor, \rightarrow, \neg$ and the quantifiers $\exists, \forall$.

Possible requirements on substitutions:

A substitutions $\sigma$ is a map from $\mathcal{F}_\mathcal{L}$ to $\mathcal{F}_\mathcal{L}$ that commutes with the connectives and quantifiers and such that . . .

- $\sigma(P(t_1, \ldots, t_n)) = \sigma(P(x_1, \ldots, x_n))[t_1/x_1, \ldots, t_n/x_n]$;
- other requirements?
**Thm** The Skolem rule $\exists x A(x, fx)/\exists x \forall y A(x, y)$ ($f$ fresh) is admissible in classical predicate logic.

**Thm** (Avigad ’03)
If a theory can code finite functions, then the Skolem rule cannot shorten proofs more than polynomially.

**Thm** (Baaz & Hetzl & Weller ’12)
In the setting of sequent calculi and cut-free proofs, the Skolem rule exponentially shortens proofs.
Heyting Arithmetic

**Thm (Iemhoff & Visser ’01)**
The Visser rules form a basis for the propositional admissible rules of Heyting Arithmetic.

**Thm (Visser ’99)**
Admissibility in Heyting Arithmetic is $\Pi_2$-complete.

**Ex** An infinite admissible rule for Heyting Arithmetic:

$$
\frac{\Gamma \Rightarrow \exists x A x}{\{ \Gamma \Rightarrow A t \mid t \text{ a term} \} \cup \{ \Gamma \Rightarrow A \mid A \in \Gamma^a \}} \quad (\Gamma \text{ implications only})
$$
Below transitivity

Thm (Jeřábek ’11) K has nullary unification.

$p \rightarrow \square p$ is admissibly saturated but not projective (not exact).
It’s a small world?

All these other logics?
Why

Provide meta-mathematical reasons for the admissibility of certain rules.