A lattice of subminimal logics of negation

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Minimal propositional calculus (MPC) is the system obtained from the positive fragment of intuitionistic propositional calculus (equivalently, positive logic [12]) by adding a unary negation operator satisfying the so-called principle of contradiction. This system was introduced by Johansson in 1937 [10] (even before, by Kolmogorov [11]) by discarding *ex falso quodlibet* from the standard axioms for intuitionistic logic.

The aim of this work is to focus on the bounded lattice of propositional logical systems arising from the language of minimal logic and obtained by weakening the requirements for the negation operator in a 'maximal way'. More precisely, the bottom element of this lattice of logics is a system where the unary operator $\neg$ has no properties at all, except the property of being functional; the top element is minimal logic. We use the term *N-logic* to denote an arbitrary logical system in this lattice. The setting is paraconsistent, in the sense that contradictory theories do not necessarily contain all formulas.

Some of these subsystems of intuitionistic logic have been studied in [7], with focus on their syntax as well as on the corresponding relational structures (e.g., their Kripke semantics). In this abstract we take the first steps towards a uniform treatment of this family of logical systems by developing their algebraic semantics. We also introduce descriptive frames for these systems and prove that every logic in this lattice is complete with respect to these descriptive frames. These results allow us to export the techniques of [8, 9, 3] to our setting and, in particular, to prove the existence of continuum many N-logics.

Given the language of positive logic (equivalently, the language of intuitionistic logic with neither negation nor $\bot$) over countably many propositional variables, we consider the axioms of positive logic and a unary operator $\neg$ satisfying the additional axiom $(p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q)$. We call the resulting system N. We keep a fixed positive logical fragment, and we strengthen the negation operator up to reaching minimal propositional logic, which can be seen in this language as the system obtained by adding the axiom $(p \to q) \wedge (p \to \neg q) \to \neg p$ to positive logic [12]. Note that another axiomatization of minimal logic is obtained by extending N with the axiom $(p \to \neg p) \to \neg p$ [6, Proposition 1.2.5].

From an algebraic point of view, we deal with relatively pseudo-complemented lattices (which algebraically characterize positive logic [12]) equipped with a unary operation $\neg$ satisfying the equation $(x \leftrightarrow y) \to (\neg x \leftrightarrow \neg y) \approx 1$. Observe that the latter can be equivalently formulated as

$$x \wedge \neg y \approx x \wedge \neg(x \wedge y).$$

We denote the variety of these algebras as $\mathcal{NA}$ and we call these structures *N-algebras*. Using the standard argument we can show that every N-logic L is complete with respect to a variety of N-algebras in which all the theorems of L are valid. The least variety among the ones we are considering, corresponding to minimal logic, is the one of contrapositionally complemented lattices [12].

Next we discuss a uniform frame-based completeness result for every N-logic. In order to do this, we introduce the notion of *top descriptive frame*: a top descriptive frame is a quadruple

$\mathfrak{F} = \langle W, R, \mathcal{P}, N\rangle$, where $\langle W, R\rangle$ is a partial order with a top node t, the set $\mathcal{P}$ is a family of admissible upsets as in the intuitionistic case [5, 3] with the difference that the top element t must be contained in every admissible upset, and $N : \mathcal{P} \to \mathcal{P}$ is a map satisfying, for all $U, V \in \mathcal{P}$,

$$U \cap N(V) = U \cap N(U \cap V). \tag{1}$$

Observe that the notion of admissible upset in this setting excludes the empty set. The positive reducts of these frame structures are presented topologically in [2] as *pointed Esakia spaces*.

It can be proved [6] that for every N-algebra there is a corresponding dual top descriptive frame, and vice versa. More precisely, given a top descriptive frame $\mathfrak{F}$, the structure

$$\mathfrak{F}^* = \langle \mathcal{P}, \cap, \cup, \to, W, N\rangle,$$

where $\to$ is the Heyting implication, is the N-algebra dual to $\mathfrak{F}$. On the other hand, the set of prime filters of any N-algebra $\mathfrak{A} = \langle A, \wedge, \vee, \to, 1, \neg\rangle$ induces a dual top descriptive frame $\mathfrak{A}_*$ defining the map N as $N(\hat{a}) = \widehat{(\neg a)}$ for every element a of $\mathfrak{A}$, where $\hat{a}$ is the upset of all prime filters containing a. Observe that the notion of prime filter in this context does not require the filter to be proper, i.e., the whole algebra A is always a prime filter, and this ensures the corresponding frame structure to have a top element.

Every N-logic L is complete with respect to the corresponding class of N-algebras. Given the afore-sketched duality between N-algebras and top descriptive frames, completeness of every N-logic with respect to the corresponding class of top descriptive frames (i.e., the ones dual to the corresponding class of N-algebras) follows easily. As in the case of Heyting algebras, for N-algebras there exists a one-to-one correspondence between congruences and filters. We can therefore characterize subdirectly irreducible N-algebras as those N-algebras containing a second greatest element, thereby obtaining a completeness result of L with respect to the class of finitely generated rooted top descriptive frames.

We conclude by proving the existence of continuum many N-logics in the interval $[\mathsf{N}, \mathsf{MPC}]$. We consider a countable family of formulas without negation that can be used to define independent systems enhancing the basic logic N, and we adapt them to ensure the logics we obtain to be subsystems of minimal propositional logic. Given a top descriptive frame and a persistent valuation map on admissible upsets, truth of a formula $\neg\varphi$ at a node w is equivalent to w being an element of the upset $N(V(\varphi))$. In what follows, we call a *top frame* a partial order with a top node; following [4], we call *top model* an intuitionistic Kripke model whose underlying frame is a top frame and such that every propositional variable is true at the top node.

Let $\Delta = \{\mathfrak{F}_n : n \in \omega\}$ be an infinite set of finite rooted top frames (see Figure 1), which forms an antichain with respect to the frame order [3] $\leq$ defined by: $\mathfrak{F} \leq \mathfrak{G}$ if and only if $\mathfrak{F}$ is an order-preserving image of a generated subframe of $\mathfrak{G}$ ([1, Lemma 6.12]; in fact, the antichain of [1] does not contain top frames, but it is easy to see that adding the top nodes still makes the sequence Δ a $\leq$-antichain). Recall (e.g., [5, 4]) that a positive morphism is a partial p-morphism f such that $\mathrm{dom}(f)$ is a downset, and consider the relation $\preceq$ defined by: $\mathfrak{F} \preceq \mathfrak{G}$ if and only if $\mathfrak{F}$ is an image, via a positive morphism, of a generated subframe of $\mathfrak{G}$. Now observe that, if $\mathfrak{F} \preceq \mathfrak{G}$ and $\mathfrak{F}$ is a top frame, then $\mathfrak{F} \leq \mathfrak{G}$. Indeed, assume $\mathfrak{F} \preceq \mathfrak{G}$ and let f be a partial p-morphism from a generated subframe $\mathfrak{G}'$ of $\mathfrak{G}$ onto $\mathfrak{F}$. Extending f by mapping all the points of $\mathfrak{G}' \setminus \mathrm{dom}(f)$ to the top node of $\mathfrak{F}$, we obtain a total order-preserving map, yielding $\mathfrak{F} \leq \mathfrak{G}$. Thus, every $\leq$-antichain is a $\preceq$-antichain, and so Δ is a $\preceq$-antichain.

Having constructed the desired $\preceq$-antichain, we will now proceed by adjusting the technique of Jankov-de Jongh formulas to obtain a continuum of logics in between N and MPC. Every finite rooted top frame $\mathfrak{F}_n$ equipped with an appropriate valuation V_n can be mapped as a top model

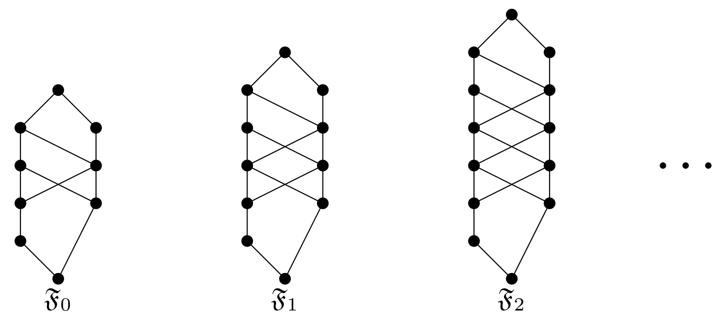

Figure 1: The antichain Δ

onto a generated submodel of $(\mathcal{U}^\star(2))^+$ via a p-morphism [4]. Consider the world $w_n \in (\mathcal{U}^\star(2))^+$ in the positive universal model corresponding to the root of $\mathfrak{F}_n$ in the sense described above. We can assume without losing generality the positive Jankov-de Jongh formula of $\mathfrak{F}_n$ to be defined as follows [4]:

$$\chi^\star(\mathfrak{F}_n) = \psi^\star_{w_n} := \varphi^\star_{w_n} \to \bigvee_{i=1}^{r} \varphi^\star_{w_{n_i}} ,$$

where $\varphi^\star_{w_n}, \varphi^\star_{w_{n_i}}$ are defined as in [4] and $w_n \prec \{w_{n_1}, \dots, w_{n_r}\}$. So, a descriptive frame $\mathfrak{G}$ refutes $\chi^\star(\mathfrak{F}_n)$ if and only if $\mathfrak{F}_n \preceq \mathfrak{G}$. As Δ is a $\preceq$-antichain, this means that, for every $n, m \in \omega$, the formula $\chi^\star(\mathfrak{F}_m)$ is valid on the frame $\mathfrak{F}_n$ if and only if $n \neq m$. In fact, it is easy to see that $\varphi^\star_{w_n}$ is satisfied at the root w_n in $(\mathcal{U}^\star(2))^+$, while none of the formulas $\varphi^\star_{w_{n_i}}$ are.

Now, we equip each frame $\mathfrak{F}_n$ with an appropriate function N_n to make it a top descriptive frame such that $N_n(\{t\}) = \{t\}$, where t is the top node of $\mathfrak{F}_n$. We denote the new family of top descriptive frames $\langle \mathfrak{F}_n, N_n \rangle$ by Δ_N. We consider a new propositional variable p and let

$$\theta(\mathfrak{F}_n) = (p \to \neg p) \wedge \varphi^\star_{w_n} \to \neg p \vee \bigvee_{i=1}^{r} \varphi^\star_{w_{n_i}} .$$

It is easy to see that, if $n \neq m$, the formula $\theta(\mathfrak{F}_n)$ is valid on the frame $\langle \mathfrak{F}_m, N_m \rangle$. On the other hand, for checking that $\langle \mathfrak{F}_n, N_n \rangle \not\models \theta(\mathfrak{F}_n)$ it is enough to consider a valuation $\tilde{V}_n$ enhancing V_n in such a way that $\tilde{V}_n(p) = \{t\}$. In this way, the root of $\mathfrak{F}_n$ under the considered valuation makes the whole antecedent of $\theta(\mathfrak{F}_n)$ true, while the consequent is not true at w_n. We note that the formulas $\theta(\mathfrak{F}_n)$ are not the Jankov-de Jongh formulas for the considered signature; in fact, $\theta(\mathfrak{F}_n)$ has the defining property of the Jankov-de Jongh formulas for the signature of positive logic with an extra addition that $\theta(\mathfrak{F}_n)$ is a theorem of MPC. The latter ensures that for each subset $\Gamma \subseteq \Delta_N$, the logic $L(\Gamma) = \mathsf{N} + \{\theta(\mathfrak{F}) : \mathfrak{F} \in \Gamma\}$ belongs to the interval $[\mathsf{N}, \mathsf{MPC}]$ i.e., $\mathsf{N} \subseteq L(\Gamma) \subseteq \mathsf{MPC}$. In particular, the logics $L(\Gamma)$ share the same positive fragment.

Finally, observe that for each pair of different subsets $\Gamma_1 \neq \Gamma_2$ of Δ_N, we have $L(\Gamma_1) \neq L(\Gamma_2)$. Indeed, without loss of generality we may assume that there is $\mathfrak{F} \in \Gamma_1$ such that $\mathfrak{F} \notin \Gamma_2$. Moreover, we have $\mathfrak{F} \not\models \theta(\mathfrak{F})$ and $\mathfrak{F} \models \theta(\mathfrak{G})$, for each $\mathfrak{G}$ in Γ_2. Therefore, there is a top descriptive frame $\mathfrak{F}$ which is an $L(\Gamma_2)$-frame and not an $L(\Gamma_1)$-frame. Since every N-logic is complete with respect to top descriptive frames, the latter entails that $L(\Gamma_1) \neq L(\Gamma_2)$. As a consequence, we obtain uncountably many distinct N-logics.

Theorem 1. *There are continuum many logics in the interval* $[\mathsf{N}, \mathsf{MPC}]$.

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