Recursive Enumerability Doesn’t Always Give a Decidable Axiomatization

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It is well-known that if a theory (deductively closed set of formulae) over a well-behaved logic (for example, classical or intuitionistic logic) is recursively enumerable (r.e.), then it has a decidable, and even a primitively recursive axiomatization [2]. This observation, known as Craig’s theorem, or Craig’s trick, is indeed very general. If we denote the deductive closure (set of theorems) for an axiomatization \( \mathcal{A} \) by \( [\mathcal{A}] \) and let \( [\mathcal{A}] \) be recursively enumerated as follows: \( \varphi_1, \varphi_2, \varphi_3, \ldots (\varphi_k = f(k), \text{where } f \text{ is a computable function}) \), then the set \( \mathcal{A}' = \{ \varphi_1, \varphi_2 \land \varphi_3, \varphi_3 \land \varphi_3, \ldots \} \) will be decidable (the decision algorithm, given a formula \( \psi \), starts enumerating \( \mathcal{A}' \), compares the elements with \( \psi \), and stops with the answer “no” when the size of the formula exceeds the size of \( \psi \): further formulae will be only bigger), and, on the other hand, \( \mathcal{A}' \) serves as an alternative axiomatization for the theory, since \( [\mathcal{A}'] = [\mathcal{A}] \).

The only thing we need from the logic for this construction to work is the following property: for any formula \( \psi \) there exists, and can be effectively constructed, an equivalent formula \( \psi' \) of greater size than \( \psi \). Then we take \( \mathcal{A}' = \{ \varphi_1, \varphi_2', \varphi_3', \ldots \} \) as the needed decidable axiomatization: since \( ' \) increases the size of formula, the \( n \)-th formula in this sequence has size at least \( n \); therefore, in our search for a given \( \psi \) in \( \mathcal{A}' \) we have to check only a

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finite number of formulae. This works even for substructural systems that
do’t enjoy $\psi \leftrightarrow \psi \land \psi$. For example, once there is an operation $\circ$ that has
a unit $1$, Craig’s theorem is valid: $A \leftrightarrow A \circ 1 = A'$.

Thus, it looks interesting to find a logic for which Craig’s theorem fails.
Of course, one could easily construct degenerate examples, like a “logic”
without any rules of inference: then $[A]$ is always $A$, and if it was r.e., but
not decidable, it doesn’t have a decidable axiomatization. So we’re seeking
for an example among interesting, useful logical systems.

And such an example exists—it is the product-free fragment of the Lamp-
bek calculus [3]. We denote this calculus by $L$ and present it here as a
Gentzen-style sequential calculus; a non-sequential (“Hilbert-style”) version
also exists [4]. Formulae of $L$ are built from a set of variables \( \text{Var} = \{p_0, p_1, p_2, p_3, \ldots\} \) using two binary connectives, $\land$ and $\lor$. Sequents are ex-
pressions of the form $A_1, \ldots, A_n \rightarrow B$, where $A_i$ and $B$ are formulae and
$n \geq 1$ (empty antecedents are not allowed). The axioms and rules of $L$ are
as follows (here capital Greek letters denote sequences of formulae):

\[
\begin{align*}
A \rightarrow A & \\
\Pi \rightarrow A \land B & \text{ where } \Pi \text{ is non-empty} \\
\Pi, A \rightarrow B & \text{ where } \Pi \text{ is non-empty} \\
\Pi \rightarrow B \lor A & \\
\Pi \rightarrow A, \Gamma, \Delta \rightarrow C & \\
\Gamma, \Pi, \Delta \rightarrow C & \text{ (cut)}
\end{align*}
\]

Let $\mathcal{A}$ be an arbitrary set of sequents. We say that a sequent $\Pi \rightarrow A$ is
derivable from $\mathcal{A}$ (denoted by $\mathcal{A} \vdash_L \Pi \rightarrow A$), if there exists a derivation tree
where inner nodes are applications of rules (including cut: in this setting it
is not eliminable), and leafs are instances of axioms or sequents from $\mathcal{A}$. The
theory axiomatized by $\mathcal{A}$ (the deductive closure of $\mathcal{A}$) is $[\mathcal{A}] = \{\Pi \rightarrow A \mid
\mathcal{A} \vdash_L \Pi \rightarrow A\}$. Clearly, if $\mathcal{A}$ is r.e., then so is $[\mathcal{A}]$. Finally, $\mathcal{A}_1$ and $\mathcal{A}_2$ are
equivalent, $\mathcal{A}_1 \approx \mathcal{A}_2$, if $[\mathcal{A}_1] = [\mathcal{A}_2]$.

**Theorem.** There exists such a recursively enumerable $\mathcal{A}$ that there is no
decidable $\mathcal{A}'$ equivalent to $\mathcal{A}$.

Let $q = p_0$ and let $\mathcal{E} = \{p_i \rightarrow q \mid i \geq 1\}$.

**Lemma.** If $\mathcal{A} \not\subseteq \mathcal{E}$ and $\mathcal{A}' \approx \mathcal{A}$, then $\mathcal{A}' \cap \mathcal{E} = \mathcal{A}$.  

2
This Lemma immediately yields our goal: if $\mathcal{A}$ is a recursively enumerable undecidable subset of $\mathcal{E}$, it gives undecidability of any $\mathcal{A}'$ equivalent to $\mathcal{A}$.

We prove the Lemma by a semantic argument, via formal language models for $\mathcal{L}$. Let $\Sigma$ be an alphabet; $\Sigma^+$ stands for the set of all non-empty words over $\Sigma$. An interpretation $w$ is a function that maps formulae of $\mathcal{L}$ to subsets of $\Sigma^+$, defined arbitrarily on variables and propagated as follows:

$$w(A \setminus B) = w(A) \setminus w(B) = \{ u \in \Sigma^+ \mid (\forall v \in w(A)) vu \in w(B) \}$$

$$w(B / A) = w(B) / w(A) = \{ u \in \Sigma^+ \mid (\forall v \in w(A)) uv \in w(B) \}$$

A sequent $A_1, \ldots, A_n \rightarrow B$ is true under interpretation $w$, if $w(A_1) \cdot \ldots \cdot w(A_n) \subseteq w(B)$, where $M \cdot N = \{ uw \mid u \in M, v \in N \}$. The calculus is sound w.r.t. this interpretation: if all formulae of $\mathcal{A}$ are true under $w$ and $\mathcal{A} \vdash_{L} \Pi \rightarrow B$, then $\Pi \rightarrow B$ is also true under $w$. (A weak completeness result, for $\mathcal{A} = \varnothing$, is shown in [1]. Here we need only soundness.)

We consider a countable alphabet, $\Sigma = \{ a_1, a_2, \ldots \}$.

First, we show that $\mathcal{A} \not\models_{L} p_i \rightarrow p_j$ for $i \neq j, i, j \geq 1$. Consider an interpretation $w_1(p_i) = \{ a_i \}$, $w_1(q) = \Sigma^+$. All sequents from $\mathcal{A}$ are true under $w_1$, while $p_i \rightarrow p_j$ isn’t. Therefore, $(p_i \rightarrow p_j) \notin \mathcal{A}'$ if $i \neq j, i, j \geq 1$.

Second, we show that $\mathcal{A} \not\models_{L} E_1 \setminus E_2 \rightarrow p_i$ and $\mathcal{A} \not\models_{L} E_2 / E_1 \rightarrow p_i$ for any $i \geq 0$ and any formulae $E_1$ and $E_2$. The counter-interpretation here is as follows: $w_2(p_i) = \{ a_i \} \cup \Sigma^{\geq 2}$, $w_2(q) = \{ a_j \mid (p_j \rightarrow q) \in \mathcal{A} \} \cup \Sigma^{\geq 2}$, where $\Sigma^{\geq 2}$ is the set of all words of length at least 2. All sequents from $\mathcal{A}$ are true under $w_2$. By induction on $A$ we show that $w_2(A) \supseteq \Sigma^{\geq 2}$ for any formula $A$. Then, since $wv$ is always in $\Sigma^{\geq 2} \subseteq w_2(E_2)$, we have $w_2(E_1 \setminus E_2) = w_2(E_2 / E_1) = \Sigma^+$, but $w_2(p_i)$ is not $\Sigma^+$ for any $i$ (including 0).

Third, we show that if $\mathcal{A} \vdash_{L} p_i \rightarrow q$, then $(p_i \rightarrow q) \in \mathcal{A}$. If not, then interpretation $w_2$ defined above falsifies $p_i \rightarrow q$ keeping all sequents in $\mathcal{A}$ true. This yields $\mathcal{A}' \cap \mathcal{E} \subseteq \mathcal{A}$ (since all sequents in $\mathcal{A}'$ are derivable from $\mathcal{A}$).

Finally, we establish the converse inclusion by contraposition. Let $(p_k \rightarrow q) \notin \mathcal{A}'$ and show that $(p_k \rightarrow q) \notin \mathcal{A}$. Consider the following interpretation: $w_3(p_i) = \{ a_i \} \cup \Sigma^{\geq 2}$, $w_3(q) = \{ a_j \mid (p_j \rightarrow q) \in \mathcal{A}' \} \cup \Sigma^{\geq 2}$. Evidently, $w_3$ falsifies $p_k \rightarrow q$. It remains to show that all sequents from $\mathcal{A}'$ are true under $w_3$. There are several possible cases for a sequent from $\mathcal{A}'$.

1. The sequent is of the form $A \rightarrow A$ (including $q \rightarrow q$ or $p_i \rightarrow p_i$). This is an axiom, it is true everywhere.

2. The sequent is of the form $p_i \rightarrow q$. Then it is true by definition.
3. The sequent is of the form $p_i \rightarrow p_j$, $i \neq j$, $i, j \geq 1$. Then this sequent is not derivable from $A$ (see above) and therefore cannot belong to $A'$.

4. The sequent is of the form $E_1 \setminus E_2 \rightarrow p_i$ or $E_2 / E_2 \rightarrow p_i$. Again, it couldn’t be derivable from $A$ and couldn’t belong to $A'$.

5. The sequent is of the form $A \rightarrow F_1 \setminus F_2$ or $A \rightarrow F_2 / F_1$. As for $w_2$, for $w_3$ we have $w_3(F_1 \setminus F_2) = w_3(F_2 / F_1) = \Sigma^+$. The sequent is true.

Hence, $A' \not\vDash_L p_k \rightarrow q$, therefore $(p_k \rightarrow q) \not\in A$. This finishes the proof.

Notice that this result is not at all robust: slight modifications of the calculus restore Craig’s theorem. First, actually one can increase the size of all formulae, except variables, by the following equivalences: $A / B \leftrightarrow A / ((A / B) \setminus A)$ and $B \setminus A \leftrightarrow (A / (B \setminus A)) \setminus A$. In our construction, we played on an infinite number of variables, for which such increasing is impossible. Thus, Craig’s theorem holds for any fragment of $L$ with a finite set of variables. Second, if we allow sequents with empty left-hand sides (and remove non-emptiness restrictions from the rules of $L$), we have $A \leftrightarrow (A / A) \setminus A$ for any formula $A$, which also yields Craig’s theorem.

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References


