Bi-modal Logics of Mappings*

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Plain maps. Initially, we consider the maps \( f : X \rightarrow Y \) between two sets. Without loss of generality, suppose that \( X \) and \( Y \) are disjoint sets. Consider a Kripke frame \( \mathcal{F}_f = (W_f, R_f) \), where \( W_f = X \sqcup Y \), \( R_f = f \), i.e. we say that, pair of points \( (x, y) \in W_f \times W_f \) is in the relation \( R_f \), iff \( f(x) = y \). The resulting Kripke frames are called Functional Frames. We say that the height of a frame \( \mathcal{F}_f = (W, R) \) is 2 if there exists \( w, u \in W \), such that \( uRw \) and for any triple of points \( (u, v, w) \in W \times W \times W \) either \( uRv \) or \( vRw \) fails. We say that a Kripke frame \( \mathcal{F}_f = (W, R) \) has no branching, if for any triple of points \( (u, v, w) \in W \times W \times W \) either \( uRv \) or \( uRw \) fails. Irreflexive frames of height \( \leq 2 \) are characterized by a formula \( \square \bot \), the no branching property is characterized by a formula \( \diamond p \land \diamond q \rightarrow \diamond (p \land q) \). We show that a Kripke frame is a Functional Frame iff it is irreflexive, non branching frame of height \( \leq 2 \). The mentioned two formulas define the class of Functional Frames. Denote

\[
K_f = K + (\Box \square \bot) + (\diamond p \land \diamond q \rightarrow \diamond (p \land q))
\]

Proposition 1. The modal logic \( K_f \) is sound and complete with respect to the class of Functional Frames.

We show that although the class of Functional Frames is modally definable, the subclasses of injective and surjective functional frames are not. If we extend the modal language by using four temporal operators \( \Box, \square, \Diamond \) and \( \lnot \), then the injective and surjective functional frames become definable. We interpret temporal operators as follows for a Kripke frame \( \mathcal{F} = (W, R) \) and \( w \in W \),

1. \( w \models \Box p \iff \forall u \in W, \ uRw \implies u \models p \).
2. \( w \models \square p \iff \forall u \in W, \ uRw \implies u \models p \).
3. \( \Diamond p = \lnot \Box \lnot p, \ \Diamond p = \lnot \Box \lnot p \).

We show that in the temporal language injective Functional Frames are determined by the formula

\[
p \rightarrow \Box \Box p,
\]

while surjective Function Frames are determined by the formula

\[
\Diamond \top \lor \Diamond \top.
\]

Order preserving maps. We consider the maps \( f : \mathcal{F}_1 \rightarrow \mathcal{F}_2 \) between Kripke frames \( \mathcal{F}_1 = (W_1, R_1) \) and \( \mathcal{F}_2 = (W_2, R_2) \). The Relational Functional Frame associated with \( f \) is a bi-relational frame \( f_R = (W, R, R_f) \), where \( W = W_1 \sqcup W_2 \), \( R = R_1 \sqcup R_2 \) and \( R_f = f \). We say \( xRy \) if either \( xR_1y \) or \( xR_2y \).

Note that \( (W, R_f) \) is a functional frame. In addition, the Relational Functional Frame \( f_R \) possesses the following coherence property: for any points \( x, y \in W \), if \( R_f(x) \neq \emptyset \) and \( xRy \lor yRx \), then \( R_f(y) \neq \emptyset \).

Proposition 2. A bi-relational Kripke frame \( \mathcal{F} = (W, R, R_f) \) is Relational Functional Frame if and only if \( R_f \) is irreflexive, its height is less than 3, it is non branching and \( R_f \), \( R \) have the coherence property.

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Proposition 5. Let $K$ be the topological structure. Suppose $\mathcal{R}_f$ is a Functional Frame with domain and co-domain of a Functional Continuous maps.

$a$ A. Chagrov and M. Zakharyaschev. Modal logic Functional Frame if $(X,\tau)$ is generated by $\mathcal{R}_f$ and $(\exists \tau)$.

$b$ P. Blackburn, M. de Rijke, and Y. Venema. Modal logic Modal Frame.


Frames modally and axiomatize the corresponding bi-modal logics.

Furthermore, we characterize the subclasses of continuous, open and interior Topological Functional Frames modally and axiomatize the corresponding bi-modal logics.

The following literature was used: [1], [2], [3].