Uniform Definability in Assertability Semantics

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As one of its benefits, logical semantics for natural language allows one to precisely answer questions about the expressive power of various fragments. Typically, one answers: what classes of structures can be defined by the fragment in question? Because, however, expressions have a given syntactic category and therefore semantic type, logical semanticists should be interested in more fine-grained conceptions of expressive power. In particular, which operations on the relevant classes of structures are definable?

This paper shows how the two kinds of expressive power can come apart, by studying disjunction and modals in the framework of assertability, or state-based, semantics. The particular framework has recently been used to explain various free-choice phenomena in natural language. We show that although the so-called ‘split’ disjunction allows no new structures to be defined, its operation is not definable, i.e. the connective is not uniformly definable. The result continues to hold in the presence of inquisitive disjunction and weak negation. The latter two extensions exhibit the following: there are two expressively complete languages, each containing a connective that is not uniformly definable in the other.

1 Languages

We are interested in multiple languages: let $\mathcal{L}$ be generated by the grammar

$$\varphi ::= p | \lnot \varphi | \varphi \land \varphi | \varphi \lor \varphi | \varphi \lhd \varphi | \varphi \lhd \varphi | \varphi$$

$\mathcal{L}_P$ is the language of propositional logic, i.e. generated by $\{\lnot, \lor, \land\}$. $\mathcal{L}_\Diamond$ also includes $\Diamond$, and mutatis mutandis for $\mathcal{L}_\lozenge$. $\mathcal{L}_P, \mathcal{L}_\Diamond, \mathcal{L}_\lozenge, \mathcal{L}_\lhd, \mathcal{L}_\lhd$ is the language generated by $\{\lnot, \land, \lor, \Diamond, \lozenge\}$. For a given language $\mathcal{L}$, let $\mathcal{L}^-$ be that language without the $\lor$ symbol.

2 Assertability Semantics

Definition 1 (Hawke and Steinert-Threlkeld [2016]). Let $W$ be a space of worlds, and $V$ a valuation function assigning subsets of $W$ to proposition letters. $s$ is an arbitrary subset of $W$. We interpret $\mathcal{L}_\Diamond$ as follows.

- $s \models^\Diamond p$ iff $s \subseteq V(p)$
- $s \models^\Diamond \lnot \varphi$ iff for every $w \in s$, $\{w\} \not\models^\Diamond \varphi$
- $s \models^\Diamond \varphi \land \psi$ iff $s \models^\Diamond \varphi$ and $s \models^\Diamond \psi$
- $s \models^\Diamond \varphi \lor \psi$ iff $s_1 \models^\Diamond \varphi$ and $s_2 \models^\Diamond \psi$ for some $s_1, s_2$ such that $s = s_1 \cup s_2$
- $s \models^\Diamond \Diamond \varphi$ iff for some $w \in s$, $\{w\} \models^\Diamond \varphi$

We write $\Gamma \models^\Diamond \varphi$ iff for every $W, V, s$, if $s \models^\Diamond \gamma$ for every $\gamma \in \Gamma$, then $s \models^\Diamond \varphi$; and $\varphi \equiv^\Diamond \psi$ iff $\{\varphi\} \models^\Diamond \psi$ and $\{\psi\} \models^\Diamond \varphi$. Let $[\varphi] = \{s : s \models^\Diamond \varphi\}$.

Fact 1. For every $\varphi \in \mathcal{L}_P$, (i) $s \models^\Diamond \varphi$ iff $\{w\} \models^\Diamond \varphi$ for every $w \in s$; (ii) $\{w\} \models^\Diamond \varphi$ iff $v^*_w(\varphi) = 1$ where $v^*_w$ is the classical propositional extension of the valuation given by $v_w(p) = 1$ iff $w \in V(p)$.

3 Definability and Uniform Definability

In this section, we show that disjunction is definable in terms of the other connectives in the following sense: for every formula in the language including disjunction ($\mathcal{L}_\Diamond$), there is a formula in the language without disjunction ($\mathcal{L}^-_\Diamond$) which is equivalent to it. The proof of this result uses a normal form theorem, which will be the main result of this section.

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1 See Hawke and Steinert-Threlkeld [2016], Aloni [2016].
Definition 2. We simultaneously define two translations $\langle \cdot \rangle_b : \mathcal{L}_\Diamond \rightarrow \mathcal{L}_P$ and $\langle \cdot \rangle_a : \mathcal{L}_\Diamond \rightarrow \mathcal{P}(\mathcal{L}_P)$:

\[
\begin{align*}
(p)_b &= p \\
(\neg p)_b &= \neg \left( (p)_b \land \bigwedge (p)_a \right) \\
(p \land q)_b &= (p)_b \land (q)_b \\
(p \lor q)_b &= (p)_b \lor (q)_b \\
(\Diamond p)_b &= T \\
(\forall \Diamond a \in (p))_b &= \emptyset \\
(\forall \Diamond a \in (p))_a &= (p)_a \\
(\Diamond (\forall a \in (p)))_a &= \{ (\forall a \in (p)) \} \\
\end{align*}
\]

Theorem 1 (Normal Form). For every $\varphi \in \mathcal{L}$, there is a $\varphi^* \in \mathcal{L}_\Diamond^*$ such that $\varphi \equiv \varphi^*$.

Proposition 1. For every $\varphi \in \mathcal{L}_\Diamond$, there is a $\varphi^* \in \mathcal{L}_\Diamond^*$ such that $\varphi \equiv \varphi^*$.

Proof. By Theorem 1, $\varphi \equiv (\varphi)_b \land \{ (\Diamond a) : a \in (\varphi)_a \}$. By the two parts of Fact 1, the propositional formulas $(\varphi)_b$ and all of the formulas $\varphi_a$ can be replaced by disjunction-free formulas while preserving equivalence.

In other words, Proposition 1 says that $\lor$ is definable in terms of $\{ \neg, \land, \Diamond \}$. It turns out, however, that for given formulas $\varphi$, $\psi$ with a disjunction, the equivalent formulas without the disjunction may bear no resemblance to each other. For example, $p \lor q$ is equivalent to $\neg (\neg p \land \neg q)$, while $\Diamond p \lor \Diamond q$ is equivalent to $\Diamond p \land \Diamond q$, but not to the corresponding de-Morgan formula. In fact, this situation is in a sense unavoidable. We will show that there is no uniform way of defining disjunction in terms of the other connectives. For now, we introduce the concept of uniform definability, before introducing the necessary machinery to prove that $\lor$ is not uniformly definable.

Definition 3 (Uniform Definability). Let $\mathcal{L}_1, \mathcal{L}_2$ be two languages interpreted in the same class of models. An $n$-ary connective $\ast$ in $\mathcal{L}_1$ is uniformly definable in $\mathcal{L}_2$ iff there is a formula $\varphi_{\ast} [r_1, \ldots, r_n]$, where $r_i$ are distinguished proposition letters, such that for all $\psi_1, \ldots, \psi_n$,

\[
\ast (\psi_1, \ldots, \psi_n) \equiv \varphi_{\ast} [r_1/\psi_1, \ldots, r_n/\psi_n].
\]

4 Directionality Functions

Having shown that $\lor$ is definable, we now show that it is not uniformly definable. The proof hinges on the way in which the various connectives interact with formulas that are upward-closed. Note that the modals let us define such sets of sets, which are not definable in standard inquisitive semantics or in dependence logic.

Definition 4. Let $X$ be a set of sets of worlds. $X$ is upward-closed $\downarrow X$ is $\perp$ iff if $s \subseteq t$ and $t \subseteq s$, then $t \in X$. $X$ is upward-closed $\downarrow X$ is $\perp$ iff if $s \in X$ and $s \subseteq t$, then $t \in X$. We say that $X$ is at most downward $\downarrow X$ is $\perp$ iff $X$ is neither $\perp$ nor $\downarrow$. For a formula $\varphi$, we say that $\varphi$ is $\perp$ (resp. $\downarrow$) iff $[\varphi]$ is $\perp$ (resp. $\downarrow$).

Call $D := \{ \top, \bot, \perp, \top \}$ the set of directions that a set of sets can have.\footnote{There is one subtlety: a formula can be both $\top$ and $\perp$, so really the functions should take on a fourth value, $\downarrow$, meaning that they are both. Of course, the only formulas that are $\downarrow$ are trivial, in the sense of having the property that $[\varphi] = P(W)$. Because of this, we omit it from subsequent discussion.} Variables like $d_i$ will range over this set. $d(X)$ will denote the direction of a set of sets $X$, with $d(\varphi)$ being shorthand for $d([\varphi])$. Our proof will show that $\lor$ acts on the set of directions differently than any formula without it. To get a taste for how this will go, we note a simple fact.

Fact 2. If $\varphi$ is $\top$, then $\varphi \lor \psi$ is $\top$.

Proof. Suppose $\varphi$ is $\top$ and let $s$ be such that $s \models \varphi \lor \psi$ and let $t \supseteq s$. We have that $s_1 \models \varphi$ and $s_2 \models \psi$ for some $s_1, s_2$ such that $s = s_1 \cup s_2$. Because $\varphi$ is $\top$, it follows that $s_1 \cup (t \setminus s) \models \varphi$. Since $s_1 \cup (t \setminus s) \cup s_2 = s \cup t \setminus s = t$, we have that $t \models \varphi \lor \psi$.

Now, each $n$-ary connective gives rise to a directionality function, specifying the direction of a formula with the connective at its outermost when its $n$ arguments have a given direction.

Proposition 2. The directionality functions $F_a$ for the connectives $\ast \in \{ \neg, \land, \lor, \Diamond \}$ are as specified in the following tables:

Definition 5. Let $\varphi [r_1, \ldots, r_n]$ be a formula with $n$ distinguished variables. The function $F_\varphi : D^n \rightarrow D$ is defined as follows: $F_\varphi(d) = d$ iff for all formulas $\varphi_1, \ldots, \varphi_n$ such that $d = \langle d(\varphi_1), \ldots, d(\varphi_n) \rangle$, $d (\varphi [r_1/\varphi_1, \ldots, r_n/\varphi_n]) = d$.

Fact 3. Let $\ast$ be an $n$-ary connective. If $\varphi_a [r_1, \ldots, r_n]$ uniformly defines $\ast$, then $F_\varphi = F_a$.

Definition 6. Let $F : D^n \rightarrow D$ be a directionality function and $d \in D$. 
Table 1: Directionality functions.

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>$\neg\varphi$</th>
<th>$\land$</th>
<th>$\uparrow \sim \downarrow$</th>
<th>$\lor$</th>
<th>$\uparrow \sim \downarrow$</th>
<th>$\varphi$</th>
<th>$\Diamond\varphi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\uparrow \downarrow$</td>
<td>$\uparrow \downarrow$</td>
<td>$\uparrow \sim \downarrow$</td>
<td>$\uparrow \uparrow \uparrow \uparrow$</td>
<td>$\uparrow \sim \uparrow$</td>
<td>$\sim \sim \sim \downarrow$</td>
<td>$\downarrow \uparrow \sim \downarrow$</td>
<td>$\uparrow \uparrow$</td>
</tr>
</tbody>
</table>

- $F$ is $d$-enforcing iff $F(\vec{d}) = \vec{d}$ for every $\vec{d} \in D^e$.
- $F$ is $d$-promoting iff $F(\vec{d}) = \vec{d}$ for every $\vec{d}$ such that $\vec{d}_i = d$ for some $i \leq m$.

We say a connective $*$ is $d$-enforcing (resp. $d$-promoting) if $F_*$ is. So, for example: $\Diamond$ is $\uparrow$-enforcing and $\lor$ is $\uparrow$-promoting.

**Theorem 2.** For every formula $\varphi[r_1, \ldots, r_n]$ in $\mathcal{L}_\Diamond$, $F_\varphi$ has the following property:

($*$) If $F_\varphi$ is $\uparrow$-promoting, then $F_\varphi$ is $\uparrow$-enforcing.

**Theorem 3.** $\lor$ is not uniformly definable in $\mathcal{L}_\Diamond$.

**Proof.** Let $\varphi[r_1, r_2]$ be an arbitrary formula without $\lor$. By Theorem 2, $F_\varphi$ satisfies ($*$). But $F_\lor$ is $\uparrow$-promoting without being $\uparrow$-enforcing, so it does not satisfy ($*$). So, $F_\varphi \neq F_\lor$. By Fact 3, $\varphi$ does not uniformly define $\lor$. □

## 5 Adding Inquisitive Disjunction

We now consider the language $\mathcal{L}_\Diamond, \lor$. We augment Definition 1 with the clause from inquisitive semantics:

$$s \Vdash \varphi \lor \psi \quad \text{iff} \quad s \Vdash \varphi \quad \text{or} \quad s \Vdash \psi$$

In this section, we show that $\lor$ is once again definable, but not uniformly definable. The results use the same methods as before: for definability, a normal form and for lack of uniform definability, directionality functions.

**Fact 4.** $\neg(\varphi \lor \psi) \equiv \neg\varphi \land \neg\psi$

**Theorem 4 (Normal Form for $\mathcal{L}_\Diamond, \lor$).** Every formula $\varphi \in \mathcal{L}_\Diamond, \lor$ is equivalent to a formula $\varphi^*$ of the form

$$\forall_i \varphi_i^0 \land \bigwedge_j \Diamond\varphi_j^i$$

where all of the $\varphi_j^i \in \mathcal{L}_P$.

**Proposition 3.** The directionality function $F_\varphi$ is given by the following table:

<table>
<thead>
<tr>
<th>$\lor$</th>
<th>$\uparrow \sim \downarrow$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\uparrow \uparrow \sim \downarrow$</td>
<td>$\sim \sim \sim \downarrow$</td>
</tr>
</tbody>
</table>

**Corollary 1.** $\lor$ is not uniformly definable in $\mathcal{L}_\Diamond, \lor$.

**Proof.** Because $\lor$ has the same directionality function as $\land$, the proof of Theorem 3 carries over to the present case. □

## 6 Expressive Completeness

Here, we show that $\mathcal{L}_\varphi, \Diamond$ can define any set of sets of worlds based on a finite set of proposition letters. Given a set $S$ of worlds and a set $P$ of proposition letters, let the restriction of $S$ to $P$ be given by $S \upharpoonright P := S / \equiv_P$ where

$$w \equiv_P w' \iff w \in V(p) \iff w' \in V(p) \text{ for every } p \in P$$

In other words, the restriction to $P$ is the quotient of $S$ by the equivalence relation that makes two worlds equivalent if they agree on all of the proposition letters in $P$. For $X$ a set of sets of worlds, let $X \upharpoonright P := \{ s \upharpoonright P \mid s \in X \}$.

**Definition 7.** A language $\mathcal{L}$ is expressively complete iff: for every finite set of proposition letters $P$ and every set of sets of worlds $X$, there is a formula $\varphi \in \mathcal{L}$ such that $[\varphi] \upharpoonright P = X \upharpoonright P$.

**Theorem 5.** $\mathcal{L}_\varphi, \Diamond$ is expressively complete.
7 Weak Negation

Finally, we consider weak negation, as studied in the setting of inquisitive semantics by Punčochář [2015]. In particular, add a unary connective $\neg$ with the following clause:

$$s \models \neg \varphi \iff s \not\models \varphi$$

**Proposition 4.** $\Box, \Diamond$ are all uniformly definable in terms of $\{\neg, \land, -\}$.

**Proof.** The reader can check that:

$$\begin{align*}
\varphi \not\bowtie \psi & \equiv - (\neg \varphi \land \neg \psi) \\
\Diamond \varphi & \equiv \neg \neg \varphi \\
\Box \varphi & \equiv \neg \neg \neg \neg \neg \varphi
\end{align*}$$

**Theorem 6** (Punčochář [2015]). $\{\neg, \land, -\}$ is expressively complete.

**Corollary 2.** $\lor$ is definable in terms of $\{\neg, \land, -\}$.

**Theorem 7.** $\lor$ is not uniformly definable in terms of $\{\neg, \land, -\}$.

**Proof.** First, note that $\neg$ has the directionality function in Table 2. Now, $F_\neg$ is not $\uparrow$-promoting, so it trivially satisfies (*). Thus, the proof technique used for Theorem 2 and Theorem 3 carry over to this setting.

$$\begin{array}{c|c|c|c|c|c|c}
\varphi & -\varphi & & & \\
\uparrow & \downarrow & & & \\
\sim & \sim & & & \\
\downarrow & \uparrow & & & \\
\end{array}$$

Table 2: Directionality function for weak negation.

**Theorem 8.** $\neg$ is not uniformly definable in terms of $\{\neg, \land, \lor, \Box, \Diamond\}$.

**Proof Sketch.** Say that a unary connective $*$ is toggling iff $F_*$($\uparrow$) = $\downarrow$ and vice versa. So, $\neg$ is toggling. An induction shows that for every formula $\varphi$ in $\{\neg, \land, \lor, \Box, \Diamond\}$, $F_\varphi$ is not toggling. By Fact 3, no such formula uniformly defines $\neg$. 

So we have two expressively complete languages. It follows that each can define all of the connectives of the other. Nevertheless, we have now seen that each language has a connective that is not uniformly definable in the other language.

8 Related Work

To conclude, we note that the distinction between definability and uniform definability also arises in inquisitive semantics and propositional dependence logic. In those settings, however, the distinction has different sources: inquisitiveness and dependence, respectively. Both logics are expressively complete for the class of downward-closed sets of sets: they can define all and only such sets. The present work shows that the lack of uniform definability for definable connectives can have a different source – the presence of upward-closed formulas – and can arise as well for expressively complete languages. Because the methods used here crucially exploit such formulas, they do not apply to the following open question: is $\lor$ uniformly definable in terms of $\{\neg, \land, \lor\}$?

References


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3 See Ciardelli [2009, 2016].

4 See Yang [2017], Yang and Väänänen [2017].