Computing Spectra of Finite Gödel Algebras through Finite Forests

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Abstract. We present two different procedures to obtain free and fine spectra of finite Gödel Algebras.

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Given a class of structures $\mathcal{C}$ and a natural number $k \geq 1$, the finite counterparts of the Shelah's classification in classical Model Theory can be given by the following problems ([7]):

- *Spectrum*: counting the number of k-element structures in $\mathcal{C}$;
- *Fine Spectrum*: counting the number of non-isomorphic k-element structures in $\mathcal{C}$;
- *Free Spectrum*: counting the elements of the free k-generated algebra in $\mathcal{C}$ (when $\mathcal{C}$ is variety of algebras).

The variety of Gödel algebras is obtained by adding the prelinearity equation to the class of Heyting algebras. Gödel algebras are the algebraic semantics of Gödel logic, a non-classical logic whose studies date back to Gödel [8] and Dummett [3]. Indeed, Gödel logic can be obtained by adding the prelinearity axiom to Intuitionistic logic. Furthermore, Gödel logic is one of the three major (many-valued) logics in Hajek's framework of Basic Logic, that is the logic of all continuous t-norms and their residua [5].

Given a finite Gödel algebra $\mathbf{A}$, the set of prime filters of $\mathbf{A}$ ordered by reverse inclusion forms a *forest* [1]. Viceversa, given a forest $\mathcal{F}$, the collection of all subforests of $\mathcal{F}$, equipped with properly defined operations, is a finite Gödel algebra. This construction is functorial, meaning that it can be extended to obtain a dual equivalence between the category of finite Gödel algebras and their homomorphisms, and the category of finite forests and open maps [2].

[1] A *forest* is a poset F where the downset of every element is totally ordered. Every downset of F is itself a forest that we call *subforest* of F.

[2] An order-preserving map between forests is *open* (or is a *p-morphism*) when it preserves downsets.

The above duality is an adaptation of the Esakia duality [4] between Heyting algebras and their homomorphisms, and posets and order-preserving open maps, to the case of finite Gödel algebras (see [2, 1] for details and proofs).

In this talk we exploit the category of forests to solve two of the above mentioned Spectra problems when $\mathcal{C}$ is the variety of Gödel algebras $\mathbb{G}$, namely the *Free Spectrum* and the *Fine Spectrum* problem.

Solutions to the *Free Spectrum* problem for $\mathbb{G}$ can be easily found in literature. Indeed, already in 1969 Horn has obtained a recurrence formula to compute the cardinalities of free k-generated Gödel algebras [6]. Another solution to this problem can be achieved by restating the Horn's recurrence in terms of finite forests [2].

Conversely, to the best of our knowledge, the *Fine Spectrum* problem for $\mathbb{G}$ has never been considered before. We introduce an algorithm that given a natural number $k \geq 1$, it generates a set of forests S_k such that for every $\mathcal{F} \in S_k$ the number of subforests of $\mathcal{F}$ is exactly k. That is, given a finite cardinal k we can build the set of finite Gödel algebras with k elements, solving in this way the *Fine Spectrum* problem for $\mathbb{G}$.

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