Flat Polygonal Logics in \( d \)-Semantics

David Gabelaia, Mamuka Jibladze, Evgeny Kuznetsov, and Levan Uridia

TSU A. Razmadze Mathematical Institute

1 Introduction

This paper is a natural follow up of a series of papers on polyhedral semantics for modal and intermediate logics. This research area became actively investigated in recent years by collaborating groups centered in Amsterdam, Milan and Tbilisi \([2, 4, 5]\). The main distinction of polyhedral semantics from standard topological semantics is in restricting valuation functions to range over polyhedral subsets of the relevant space endowed with some kind of linearity structure – polyhedra in Euclidean spaces being the prime examples.

Let \( B_n \) be the Boolean subalgebra of the full powerset Boolean algebra \( \mathcal{P}(\mathbb{R}^n) \) of all subsets of \( \mathbb{R}^n \) generated by (either open or closed) halfspaces. Elements of \( B_n \) are called polyhedral sets. \( B_n \) turns out to be closed under the topological closure or derived set operators. It is well known that these operators serve as a basis for two distinct topological interpretations for modal language. More widely known \( C \)-semantics treats modality (the diamond) as the closure operator of a topological space. In algebraic terms this amounts to dealing with the classes of closure algebras. Lesser known \( d \)-semantics interprets the modal diamond as the derivative operator of a topological space. Algebraically this amounts to the investigation of the classes of derivative algebras. It is straightforward that \( B_n \) can be treated as a closure algebra, since the closure \( C(P) \) of a polyhedron \( P \) is again a polyhedron \( C(P) \in B_n \). In a similar way, the set \( d(P) \) of all limit points of a polyhedron \( P \) is a polyhedron. To make a clear distinction between the resulting modal algebras, by \( B_n \) we denote the closure algebra, while by \( B_n^d \) we denote the derivative algebra of all subpolyhedra of \( \mathbb{R}^n \). The \( C \)-logic \( \text{Log}(B_2) \) of two-dimensional polyhedra is studied and axiomatized in \([4]\) while the \( d \)-logic of two-dimensional polyhedra \( \text{Log}(B_2^d) \) is studied and axiomatized in \([5]\). In the present contribution we are interested in polyhedral \( d \)-logics.

Recall that the modal logic \( K4 = K + \Box p \rightarrow \Box \Box p \) is the logic of transitive Kripke frames. The logic \( K4,Grz \) is defined as \( K4 + \Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow \Box p \). It turns out that \( B_d \) is a locally finite \( K4,Grz \)-algebra.

For a relativization of \( B_d \) to a polyhedral set \( P \in B_d \) we will use notation \( P^+ \). We consider polyhedral \( d \)-logics – logics \( \text{Log}\{P_i^+ | i \in I\} \), generated by some family \( \{P_i\}_{i \in I} \) of polyhedra \( P_i \in B_d^i \). Since each \( B_d^i \) is a locally finite \( K4,Grz \)-algebra, polyhedral \( d \)-logics are extensions of \( K4,Grz \) and each one of them has the finite model property.

In the current work we axiomatize the largest polyhedral \( d \)-logic, i.e. the \( d \)-logic of all polyhedra. We also study in details the \( d \)-logics of polyhedra of dimension 2 or less. In particular we fully characterize flat polygonal \( d \)-logics, that is 2-dimensional \( d \)-logics generated by any class of polygons \( P_i \in B_d^i \) embeddable inside the 2-dimensional plane \( \mathbb{R}^2 \).

2 Polyhedral \( d \)-Logics

Polyhedral \( d \)-logics are generated by algebras of type \( P_i^+ \) where \( P_i \in B_d^i \) is a polyhedron. Each \( P_i^+ \) is of finite height and hence, locally finite \([3]\). This has to do with the geometric dimension of \( P \) being finite. It follows that polyhedral \( d \)-logics enjoy the finite model property and their study
can be reduced to the study of the corresponding finite Kripke frames. Since \( P^+ \) is always a \( K4.Grz \)-algebra, its finite Kripke frames are finite weak partial orders i.e. frames \((W, R)\) such that the reflexive closure \( R^\ast \) of \( R \) is a partial order. We call such frames \( w \)-posets. Note that \( w \)-posets are transitive and antisymmetric.

Each polyhedral \( d \)-logic \( L \) has well-defined dimension \( \dim L \): it is either the smallest \( d \) for which \( L \) forbids the \((d + 1)\)-element reflexive chain, or infinity, if such a \( d \) does not exist. This happens to coincide with the maximum of the geometric dimensions of the polyhedra \( P \) which validate \( L \). The polyhedral \( d \)-logics of finite dimension are of finite height and hence, locally finite. In the next theorem we give the axiomatization of the logic of all polyhedra in \( d \)-semantics.

**Theorem 1.** The \( d \)-logic of all polyhedra is \( K4.Grz + \square(\square p \rightarrow p) = K4.Grz + \sigma(\uparrow) + \sigma(\downarrow) \)

Here and in what follows by \( \sigma(\tilde{\delta}) \) we denote the subframe axiom of the \( w \)-poset \( \tilde{\delta} \) [6]. The depiction of \( w \)-posets follows the convention of denoting reflexive points by white circles and the irreflexive points by filled black circles.

In the following theorem we focus on the fixed dimension \( n \) and characterise/axiomatize the minimal and maximal \( d \)-logics of dim \( n \) polyhedra.

**Theorem 2.** Let \( L \) be a polyhedral \( d \)-logic of dim \( n \). Then \( K4.Grz_n \subseteq L \subseteq PL^d_n \), where:

1. Maximal polyhedral \( d \)-logic of dim \( n \) is \( PL^d_n = \log(\mathcal{B}^n_d) \)

2. Minimal polyhedral \( d \)-logic of dim \( n \) is \( K4.Grz_n := K4.Grz + \square(\square p \rightarrow p) + \sigma\left(\begin{array}{c} \mathcal{G}^n \vspace{1mm} \\
\mathcal{G} \end{array}\right)\)

Where \( \sigma\left(\begin{array}{c} \mathcal{G}^n \vspace{1mm} \\
\mathcal{G} \end{array}\right) \) is the subframe axiom forbidding the \((n + 1)\)-element reflexive chain.

There is a single polyhedral \( d \)-logic of dim \( 0 \) – the logic of one irreflexive point characterised by axiom \( \Box \bot \). Let us denote by \( \tilde{\delta}^+_n \) the rooted \( w \)-poset of height 2 with irreflexive root and \( n \)-many maximal reflexive points.

**Theorem 3.** Polyhedral \( d \)-logics of dim 1 form a countable chain (under inclusion) between \( K4.Grz_1 \) and \( PL^d_1 \) which is presented as follows:

\[
K4.Grz_1 \subseteq \cdots \subseteq \log(\tilde{\delta}^+_1) \subseteq \cdots \subseteq \log(\tilde{\delta}^+_2).
\]

We now turn to flat polyhedra – those dim \( n \) polyhedra that are embedded into the ambient Euclidean space \( \mathbb{R}^n \) of the same dimension. The relevant algebraic notion is that of relativization, while the relevant modal notion is that of downward subframization [6], [1]. Call the polyhedral \( d \)-logic \( L \) of dim \( n \) flat iff \( L \) is complete wrt some class \( (P^+_i)_{\in I} \) of polyhedral derivative algebras such that \( P_i \in \mathcal{B}^n_d \) are polyhedra of dim \( n \) inside \( \mathbb{R}^n \) for all \( i \in I \).

**Theorem 4.** The least flat polyhedral \( d \)-logic \( \text{Flat}^d_n \) of dim \( n \) is the downward subframization of \( PL^d_n \).

Our main results concern the flat \( d \)-logics of dim \( 2 \) – we call them Flat Polygonal \( d \)-Logics. By definition, such logics are generated by a family of relativizations of \( \mathcal{B}^2_d \). In other words, flat polygonal logics are complete wrt some class \( (P^+_i)_{\in I} \) where each \( P_i \) is a flat polygon – a polygonal subset of the Euclidean plane \( \mathbb{R}^2 \). We will give a full characterization of flat polygonal \( d \)-logics, using an explicit collection of Jankov-Fine axioms for certain finite \( w \)-posets. It turns out that \( \text{Flat}^d_2 \) is the logic of finite \( K4.Grz_2 \)-frames which are not up-reducible to the poset \( \bigcup \)

\[
\text{Flat}^d_2 \text{-Semantics}
\]
Theorem 5. \( \text{Flat}^d_2 = \text{K4.Grz}_2 + \chi \left( \circ \downarrow \circ \right) \) where \( \chi \left( \circ \downarrow \circ \right) \) is the Jankov-Fine axiom forbidding the reflexive 3-fork (as an up-reduction). Flat polygonal logics are all in the interval \([\text{Flat}^d_2, \text{PL}^d_2]\). 

To describe the flat polygonal logics occurring between \( \text{Flat}^d_2 \) and \( \text{PL}^d_2 \), we introduce w-posets \( \delta_{m,n}^\ast \) with irreflexive root depicted below that are ordered by reducibility – \( \delta \) is reducible to \( \delta' \) if there exists an onto p-morphism from \( \delta \) to \( \delta' \). The poset of these frames is depicted on Figure 2.

The reducibility among \( \delta_{m,n}^\ast \) can be described as follows: \( \delta_{m,n}^\ast \) reduces to \( \delta_{m',n'}^\ast \) iff \( m + n \geq m' + n' \) and \( m \geq m' \). Denote the poset of these frames by \( Q \).

Lemma 6. The dual poset of \( Q \) is a well partial order, i.e. \( Q \) contains neither infinite strictly ascending chains, nor infinite antichains.

For every antichain \( \alpha \) in \( Q \) the corresponding \( d \)-logic \( L_\alpha \) is obtained by adding to \( \text{Flat}^d_2 \) the Jankov-Fine axioms \( \chi(\delta_{m,n}^\ast) \) for each \( \delta_{m,n}^\ast \in \alpha \). It is not difficult to see, that \( L_\alpha \subseteq L_\beta \) if \( \alpha \subseteq \downarrow \beta \). Moreover:

Theorem 7. The \( d \)-logics \( L_\alpha \), for \( \alpha \subset Q \) an antichain, are all different, and exhaust all flat polygonal \( d \)-logics, that is all polygonal \( d \)-logics between \( \text{Flat}^d_2 \) and \( \text{PL}^d_2 \).

It follows that there are only countably many flat polygonal \( d \)-logics, each of which is finitely axiomatizable and decidable. In the talk we will also present a way to describe the Kripke frames for each flat polygonal \( d \)-logic \( L \) based on the upset of \( L \)-frames inside \( Q \) and a certain operation on w-posets defining \( \text{PL}^d_2 \) and \( \text{PL}^d_4 \) – ir-crown frames [5] and \( n \)-forks with irreflexive root \( \delta_n^\ast \).

References