Analysis of the multi-modal logics for modal maps*

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Plain maps. Initially, we consider maps \( f : X \rightarrow Y \) between two sets. Suppose that \( X \) and \( Y \) are disjoint sets. consider a Kripke frame \( \mathcal{F}_f = (W_f, R_f) \), where \( W_f = X \sqcup Y \), \( R_f = f \), i.e. we say that, pair of points \( (x, y) \in W_f \times W_f \) is in the relation \( R_f \), iff \( f(x) = y \). The resulting Kripke frames are called Functional frames. We say that the height of a frame \( \mathcal{F} = (W, R) \) is 2 if there exists \( w, u \in W \), such that \( uRw \) and for any triple of distinct points \( (u, v, w) \in W \times W \times W \) either \( uRv \) or \( vRw \) fails. We say that a Kripke frame \( \mathcal{F} = (W, R) \) has no branching, if for any triple of points \( (u, v, w) \in W \times W \times W \) either \( uRv \) or \( uRw \) fails. Irreflexive frames of height \( \leq 2 \) are characterized by a formula \( \Box \Box \bot \), the no branching property is characterized by a formula \( \Diamond p \land \Diamond q \rightarrow \Diamond(p \land q) \). We show that a Kripke frame is a Functional Frame iff it is irreflexive, non branching frame of height \( \leq 2 \). The mentioned two formulas define the class of Functional Frames. Denote

\[
K_f = K + (\Box \Box \bot) + (\Diamond p \land \Diamond q \rightarrow \Diamond(p \land q))
\]

Proposition 1. The modal logic \( K_f \) is sound and complete with respect to the class of Functional Frames.

Proposition 2. \( K_f \) has the finite model property.

We show that although the class of Functional Frames is modally definable, the subclasses of injective and surjective functional frames are not. If we extend the modal language by using four temporal operators \( \Box, \Diamond, \text{and} \Diamond \), then the injective and surjective functional frames become definable. We interpret temporal operators as follows for a Kripke frame \( \mathcal{F} = (W, R) \) and \( w \in W \),

1. \( w \models \Box p \) iff \( \forall u \in W \), \( uRw \) implies \( u \models p \).
2. \( w \models \Diamond p \) iff \( \forall u \in W \), \( uRw \) implies \( u \models p \).
3. \( \Diamond p = \neg \Box \neg p, \Diamond p = \neg \Box \neg p \)

We show that in the temporal language injective Functional Frames are determined by the formula

\[
p \rightarrow \Box \Box p,
\]

while surjective Function Frames are determined by the formula

\[
\Diamond T \lor \Diamond T.
\]

Order preserving maps. We consider the maps \( f : \mathcal{F}_1 \rightarrow \mathcal{F}_2 \) between Kripke frames \( \mathcal{F}_1 = (W_1, R_1) \) and \( \mathcal{F}_2 = (W_2, R_2) \). The Relational Functional Frame (RFF for brevity) associated with \( f \) is a bi-relational frame \( f_R = (W, R, R_f) \), where \( W = W_1 \sqcup W_2 \), \( R = R_1 \sqcup R_2 \) and \( R_f = f \). We say \( xRy \) if either \( xR_1y \) or \( xR_2y \).

Note that \( (W, R_f) \) is a functional frame. In addition, the Relational Functional Frame \( f_R \) possesses the following coherence property: for any points \( x, y \in W \), if \( R_f(x) \neq \emptyset \) and \( xRy \lor yRx \), then \( R_f(y) \neq \emptyset \).

Proposition 3. A bi-relational Kripke frame \( \mathcal{F} = (W, R, R_f) \) is an RFF (Relational Functional Frame) if and only if \( R_f \) is irreflexive, its height is less than 3, it is non-branching and \( R_f \). It have the coherence property.

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Since we deal with bi-relational frames, in the syntactic signature we now have $\Box$, $\Diamond$ for $R_f$ and $\Box$, $\Diamond$ for $R$, hence we switch to bi-modal language. Here $\Box$, $\Diamond$ are interpreted as in Functional Frames and the $\Box$, $\Diamond$ are interpreted as follows

- For any formula $\varphi$ in a language, $\Box \varphi$ is satisfiable in $w \in W$ if $\varphi$ is true in all $R$-successors of $w$.
- For any formula $\varphi$ in a language, $\Diamond \varphi$ is satisfiable in $w \in W$ if there exists an $R$-successor $u \in W$ of $w$, such that $\varphi$ is satisfiable in $u$.

The Coherence Property in RFFs corresponds to the following formulas

$\Diamond \top \rightarrow \Box \Diamond \top$

$\Diamond \Diamond \top \rightarrow \Diamond \top$

We show that the class of all RFFs is modally definable in the bi-modal language. Let $K_R$ be defined as follows

$K_R = K_{\Box} \Box + (\Box \Diamond \bot) + (\Diamond p \land \Diamond q \rightarrow \Diamond (p \land q)) + (\Diamond \top \rightarrow \Box \Diamond \top) + (\Diamond \Diamond \top \rightarrow \Diamond \top)$

**Proposition 4.** Bi-modal logic $K_R$ is sound and complete with respect to the class of all Relational Functional Frames.

We now impose some natural conditions on the underlying map $f$ to see if the class of resulting RFFs is modally definable and finitely axiomatizable. Recall, that a map $f : \mathfrak{G}_1 \rightarrow \mathfrak{G}_2$ is preserving iff $wR_1 v$ implies $f(w)R_2 f(v)$ for all $w, v \in W_1$. The map $f$ is called reflecting iff for any $w \in W_1$ and $u \in W_2$, whenever $f(w)R_2 u$ holds, there exists $u' \in W_1$ such that $wR_1 u'$ and $f(u') = u$. The map $f$ is called a $p$-morphism iff it is both preserving and reflecting.

Let us adopt the following notation for the corresponding logics: Let $K_{pre}$ denote the bi-modal logic of all RFFs with $f$ preserving; let $K_{ref}$ denote the logic of all RFFs with $f$ reflecting and let $K_p$ denote the logic of all RFFs with $f$ a $p$-morphism.

**Proposition 5.** The three logics are axiomatized as follows:

$K_{pre} = K_R + (\Diamond \Diamond p \rightarrow \Diamond \Diamond p)$

$K_{ref} = K_R + (\Diamond \Diamond p \rightarrow \Diamond \Diamond p)$

$K_p = K_R + (\Diamond \Diamond p \leftrightarrow \Diamond \Diamond p)$

and define their respective classes of RFFs.

Moreover, we can show that $K_R$ and $K_{pre}$ logics are determined by their finite frames, i.e.:

**Proposition 6.** $K_R$ and $K_{pre}$ have the finite model property.

From the fmp and the finite axiomatization we conclude that $K_R$ and $K_{pre}$ logics are decidable and in fact, their decision problem turns out to be in PSPACE.

**Continuous maps.** Now instead of relations, we equip the domain and co-domain of a Functional Frame with topological structure. Suppose $f : X_1 \rightarrow X_2$ is a map between topological spaces $(X_1, \tau_1)$ and $(X_2, \tau_2)$. Let us introduce $f_\tau = (X, \tau, R_f)$ topological structure, where $X = X_1 \sqcup X_2$, $\tau$ is a topology generated by $\tau_1 \sqcup \tau_2$, and $R_f = f$. A topological structure $f_\tau = (X, \tau, R_f)$ is called a Topological Functional Frame (TFF for brevity) if $(X, R_f)$ is a Functional Frame and $R_f^{-1}(X) = \{x \in X \mid \exists y \in X \text{ with } yR_f x\}$ is clopen (simultaneously closed and open according to $\tau$). The latter condition is the Coherence Property for TFFs.

Again, due to existence of two, topological and function structures, we have two kinds of modal operators in our language, $\Box$, $\Diamond$ and $\Box$, $\Diamond$ respectively. The operator $\Diamond$ is interpreted as $f^{-1}$. The operator $\Box$ is interpreted as topological Interior operator and the operator $\Diamond$ – as topological Closure operator.
We show that the class of all TFFs (Topological Functional Frames) is modally definable. The bi-modal logic of Topological Functional Frames is denoted by $S4_R$. We axiomatize this logic as follows:

$$S4_R = K + (\Box p \rightarrow p) + (\Diamond p \rightarrow \Box \Box p).$$

**Proposition 7.** Bi-modal logic $S4_R$ is sound and complete with respect to the class of all Topological Functional Frames.

Furthermore, we characterize the subclasses of continuous, open and interior TFFs modally and axiomatize the corresponding bi-modal logics. Let us recall that a map $f : X_1 \rightarrow X_2$ is called **continuous** if the $f$-pre-images of $\tau_2$-open sets are open in $\tau_1$; the map is **open** if the images of open sets are open and the map is **interior** if it is both open and continuous.

Let us adopt the following notation for the corresponding bi-modal logics: let $S4_c$ denote the logic of all TFFs with $f$ continuous; let $S4_o$ denote the logic of all TFFs with $f$ open and let $S4_i$ denote the logic of all TFFs with $f$ interior.

**Proposition 8.** The three logics are axiomatized as follows:

$$S4_c = S4_R + (\Diamond \Diamond p \rightarrow \Diamond p)$$

$$S4_o = S4_R + (\Diamond \Box p \rightarrow \Box \Diamond p)$$

$$S4_i = S4_R + (\Box \Diamond p \leftrightarrow \Diamond \Box p)$$

and define their respective classes of TFFs.

Moreover, we show that

**Proposition 9.** The logics $S4_R$ and $S4_c$ have the finite model property.

From the fmp and the finite axiomatization we conclude that $S4_R$ and $S4_c$ logics are decidable. In fact, their decision problems are in the complexity class PSPACE.

The following literature was used:[1], [2], [3].

**References**

