The role of intuitionistic reasoning in the development of the proof mining methodology
dedicated to the memory of Anne S. Troelstra (1939-2019)

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Anne Troelstra Memorial Event, Amsterdam, March 6, 2020
Brief background on Proof Mining
In my 1990 PhD thesis (1st referee: Horst Luckhardt, 2nd referee: Anne S. Troelstra) applied ‘proof mining’ (Dana Scott) to problems in analysis.
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Analysis particularly rewarding since the appropriate choice of representations of analytical objects matters (e.g. standard representations of CSM and CTB spaces, Troelstra 1966-69).

Interesting proofs that use WKL (Troelstra, JSL 1974) but allow for a WKL-elimination: uniqueness statements. (∈∀→∀).

Carrying all this out in an extended case study: moduli and constants of strong unicity in best Chebycheff approximation (later with P. Oliva also for $L_1$-approximation).
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Proof Mining since 2000 (abstract classes of spaces)

Around 2000: started to apply proof mining in metric fixed point theory:

Let $X$ be some Banach space, $C \subseteq X$, $T: C \to C$ a selfmap that e.g. is nonexpansive:

$$\forall x, y \in C \left( \|Tx - Ty\| \leq \|x - y\| \right).$$

Consider appropriate iterations such as the Krasnoselski-Mann iteration

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Tx_n,$$

$x_0 \in C$, for suitable $(\lambda_n) \subset [0,1]$.

Under appropriate conditions $(x_n)$ converges to a fixed point of $T$.

Under much more general conditions (e.g. $(x_n)$ being bounded), one has asymptotic regularity

$$\|x_n - Tx_n\| \to 0.$$
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$$\|x_n - Tx_n\| \rightarrow 0.$$
Why rewarding for proof mining?

- $\|x_n - Tx_n\| \to 0$ has form $\forall \exists$ since $(\|x_n - Tx_n\|)$ is decreasing.

**Hence:** extractability of full rates of convergence for $\|x_n - Tx_n\| \to 0$. 

Numerous similar results not only in fixed point theory but also ergodic theory, convex optimization, nonlinear semigroup theory etc.

The finitary proof-theoretic analysis makes it easy to generalize things to geodesic settings (with L. Leuștean).

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Since 2004: rates of metastability

If \( \| x_n - T x_n \| \) is not monotone or one studies the convergence of \((x_n)\) itself, in general no computable rate of convergence possible.

Let \((x_n)\) be a Cauchy sequence in a metric space \((X, \rho)\), i.e.

\[
\forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall i, j \geq n (\rho(x_i, x_j) \leq 2^{-k}) \in \forall \exists
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is noneffectively equivalent to its Gödel functional interpretation

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\forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} \forall i, j \in [n; n + g(n)] (\rho(x_i, x_j) < 2^{-k}) \in \exists
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Herbrand normal form or metastability (Tao).
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A bound \(\Phi(k, g)\) on ‘\(\exists n\)’ in the latter formula is a rate of metastability (introduced by Kreisel in 1951 as no-counterexample interpretation).
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- Huge gap between **ideal principles** used and the **concreteness** of the theorem proven: Cauchy-property (\(\Pi^0_3\)).
- Concrete bounds numerically interesting. Often information on the **algorithmic learnability** of a rate of convergence which - if a gap condition is satisfied - yields **oscillation bounds** (Safarik/K., Avigad/Rute).
Inspiration from Troelstra’s work on Intuitionism I:

Closure under Fan Rules
(Troelstra JSL 1974,1977)
The fan rule

In his 74/77 JSL-papers, Troelstra proved among many other things:

**Theorem (Troelstra 74,77)**

Let $H^\omega$ be intuitionistic arithmetic in all types or analysis $E$-$HA^\omega$, $N$-$HA^\omega$ or $EL$. Then $H^\omega$ is closed under the fan rule, i.e.

$$H^\omega \vdash \forall f^1 \exists n^0 A(f, n) \Rightarrow H^\omega \vdash \forall g^1 \exists n^* \forall f \leq g \exists n \leq n^* A(f, n).$$
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- Troelstra 1974 uses modified realizability (with truth) and the uniform continuity of closed terms $t^2 \in T$.
- Troelstra 1977 uses a notion of **fan computability** and Troelstra/van Dalen 1988 **uniform forcing**.
Consider extensional Heyting arithmetic $E$-$HA^{\omega \cdot X} [X, d, b]$ over all finite types over the base types $\mathbb{N}, X$ where $X$ represents an abstract $b$-bounded metric space with

$$x =_X y := d_X(x, y) = \mathbb{R} 0.$$
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Let

\[
\text{CA}_\exists : \exists \Phi \forall x (\Phi(x) =_{\mathbb{N}} 0 \iff \neg A(x)),
\]

where \( x \) is an arbitrary tuple of variables of \emph{arbitrary types} and \( A \) an \emph{arbitrary formula}.

Let \( \text{AC} \) be the axiom-of-choice schema in all types.

A fan-type rule for abstract spaces

A simple version reads as follows:

**Theorem (Gerhardy/K. APAL 2006)**

Let $\rho$ (resp. $\tau$) be an arbitrary type with values in $\mathbb{N}$ (resp. in $X$). $s$ is a closed term. If (for arbitrary $A, B$)

$$E$-${\text{HA}}^{\omega,X}[X, d, b] + AC + CA \vdash \forall x^1 \forall y \leq \rho \ s(x) \forall z^\tau \ (\neg B \rightarrow \exists n^{\mathbb{N}} A)$$

then one can extract a functional $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \rightarrow \mathbb{N}$ in Gödel’s $T$ s.t.

$$\forall x^1 \forall y \leq \rho \ s(x) \forall z^\tau \exists n \leq \Phi(x, b) \ (\neg B \rightarrow A)$$

holds in any $b$-bounded metric space.
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Also for $\text{WE-HA}^\omega,X [X, d, b] + \text{M}^\omega + \text{KL}$ with $B_\forall$. 


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**Methods:** monotone versions of extensions of mr resp. Dialectica.
The usual fan rule as a special case

- The **usual fan rule** for intuitionistic systems such as E-HA$^\omega$ is the special case without classical principles such as CA$^\leftarrow$, without abstract spaces $X$ and for $\rho = 1$.
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- In 1995, I showed that the same is true if noneffective principles such as CA$\neg$ or M$^\omega + KL$ (not both!) are added.
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- The **usual fan rule** for intuitionistic systems such as E-HA°ω is the special case without classical principles such as CA¬, without abstract spaces X and for ρ = 1.
- The proof in K.1992 allowed for arbitrary types ρ and so no longer rests on continuity but on majorizability.
- In 1995, I showed that the same is true if noneffective principles such as CA¬ or Mω+KL (not both!) are added.
- Despite some efforts, I never managed to find an application for ρ > 1 in mainstream mathematics. However: there are many applications for τ = X and for τ = X^X!
Classically ($\Sigma_1^0$-LEM), the fan rule seems to fail miserably: consider

$$\forall f \in 2^\mathbb{N} \exists n \in \mathbb{N} \ \forall k \in \mathbb{N} (f(k) = 0 \rightarrow f(n) = 0).$$

However, it holds (even in the much generalized form) if $A_\exists$ is purely existential (and one weakens the extensionality axiom to a quantifier-free rule and restricts AC to quantifier-free formulas):
The fan rule from a classical point of view

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With some mild restrictions on the types, we may add dependent choice DC, where then $\Phi$ will be bar recursive.

Method: extended version of monotone Dialectica with negative translation.
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Let $\rho$ (resp. $\tau$) be an arbitrary type with values in $\mathbb{IN}$ (resp. in $X$). $s$ is a closed term. If

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Other admissible structures $X$

- Hyperbolic, CAT(0), CAT($\kappa > 0$), normed, their completions, Hilbert, uniformly convex, uniformly smooth (not: separable, strictly convex or smooth) spaces.

- Also several spaces $X_1, \ldots, X_n$ (Günzel/K.)
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- Also several spaces $X_1, \ldots, X_n$ (Günzel/K.)

- All normed structures definable in positive bounded logic, e.g. abstract $L_p$ and $C(K)$-spaces (Günzel/K. 2016).
The unbounded case

Here \( z^T \) needs to be majorizable (extending Howard’s notion: Gerhardy/K.2008): \( y, x \) functionals of types \( \rho, \hat{\rho} := \rho[\mathbb{N}/X] \) and \( a^X \)

\[
\begin{align*}
\text{\( x^\mathbb{N} \succ^a_{\mathbb{N}} y^\mathbb{N} \equiv x \geq y \)} \\
\text{\( x^\mathbb{N} \succ^a_X y^X \equiv x \geq d_X(a, y) \)).}
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Here $z^\tau$ needs to be majorizable (extending Howard’s notion: Gerhardy/K. 2008): $y, x$ functionals of types $\rho, \hat{\rho} := \rho[\mathbb{N}/X]$ and $a^X$

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x^\mathbb{N} \gtrsim^a y^\mathbb{N} \equiv x \geq y
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For complex types $\rho \rightarrow \tau$ this is extended in a hereditary fashion.
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Example:

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f^* \mathrel{\geq^a} f \equiv \forall n \in \mathbb{N}, x \in X[n \geq d_X(a, x) \to f^*(n) \geq d_X(a, f(x))].
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If $f : X \rightarrow X$ is nonexpansive (n.e.), i.e. $d_X(f(x), f(y)) \leq d_X(x, y)$. 
The unbounded case

Here $z^\tau$ needs to be majorizable (extending Howard’s notion: Gerhardy/K.2008): $y, x$ functionals of types $\rho, \hat{\rho} := \rho[\mathbb{N}/X]$ and $a^X$

$$x^\mathbb{N} \gtrsim a_{\mathbb{N}}^X y^\mathbb{N} : \equiv x \geq y$$

$$x^\mathbb{N} \gtrsim a_X^X y^X : \equiv x \geq d_X(a, y).$$

For complex types $\rho \rightarrow \tau$ this is extended in a hereditary fashion.

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Then for $d_X(a, f(a)) \leq b$ and $f^*(n) := n + b$: $f^* \gtrsim a_{X \rightarrow X}^X f$.

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x^IN \gtrsim^a IN y^IN :\equiv x \geq y \\
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In a normed setting: $a := 0_X$. 
Inspiration from Troelstra’s work on Intuitionism II:
Conservation results for the Fan Principle
(Troelstra JSL 1974)
Let EL$^+$ be elementary intuitionistic analysis plus AC$^{\mathbb{IN}, \mathbb{IN}}$ plus a continuity principle CONT$_1$. Consider FAN in the form (equivalent to Troelstra’s definition of FAN over EL$^+$)

$$\forall f \in 2^{\mathbb{IN}} \exists n \in \mathbb{IN} A(f, n) \rightarrow \exists n^* \in \mathbb{IN} \forall f \in 2^{\mathbb{IN}} \exists n \leq n^* A(f, n).$$

**Theorem (Troelstra, JSL 1974)**

EL$^+$ + FAN is conservative over HA.
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$\text{EL}^+ + \text{FAN}$ is conservative over $\text{HA}$.

Further contributions in Troelstra 1974:

- For WKL (stated for the first time in print and called KL) it is proven that $\text{EL}^c + \text{WKL}$ is conservative over $\text{PA}$. 
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In Troelstra’s 1977 Handbook article: elimination of choice sequences used to show conservativity of FAN.
Conservation of generalized fan principles

- The **generalized fan rules** (both in the semi-constructive and in the classical case) can be also stated as **implications**: **uniform boundedness principles** (K.1995 without $X$, K. 2006 with $X$).
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Even more general forms for uniform boundedness principles are studied in the context of **bounded functional interpretation** as **‘bounded collection principles’** (Engracia 2009, Ferreira 2009).
Consequences of $\exists$-UB$^X$

Over $\text{WE-PA}^{\omega, X}[X, d, b] + \text{QF-AC}$, the principle $\exists$-UB$^X$ proves the following classically in general false facts (K.2006):

- $X$ is complete.
- If $X$ is separable then it is totally bounded.
- Every function $F : X \to X$ is uniformly continuous.
- If $F : X \to \mathbb{R}$ has approximate zeros, then it has zeros.
- For $W$-hyperbolic $X$: every nonexpansive $F : X \to X$ has fixed points.
- If $X$ is strictly convex then it is uniformly convex (many more in: Gündel/K. 2016).

$\exists$-UB$^X$ serves as a constructive substitute to ultrapowers. Recently shown to allow to replace some uses of weak sequential compactness (Ferreira, Leuștean, Pinto 2019).
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Inspiration from Troelstra’s work on Intuitionism III:
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Intensional aspects of choice sequences.
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Extensionality without continuity (Leustean, Nicolae, K.)

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**Frequent use of extensionality:** \( T : C \to C \subseteq X, F(T) \) fixed point set.

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\forall x \in C, p \in F(T) \ (x =_x p \to x \in F(T))
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\]

Important in fixed point theory: mappings with Suzuki’s condition

\[
\forall x, p \in C (\|p - Tx\| \leq \mu \|p - Tp\| + \|x - y\|) \quad (\mu \geq 1).
\]

Then \( \delta_T(\varepsilon) = \varepsilon/4, \omega_T(\varepsilon) = \varepsilon/(2\mu) \). No continuity requirement!
Inspiration from Troelstra’s work on Intuitionism IV:
Restricted forms of LEM (e.g. Troelstra/van Dalen 1988)
The role of classical logic

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- Same for $\text{HA}^\omega[X, d, \ldots]+\text{AC}+\text{KL}+M^\omega$, where $M^\omega$ is Markov’s principle in all types and KL is König’s lemma (K.1995,2008).
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- Same for $\text{HA}^\omega[X, d, \ldots]+\text{AC}+\text{KL}+\text{M}_\omega$, where $\text{M}_\omega$ is Markov’s principle in all types and KL is König’s lemma (K.1995,2008).
- Naive formalization of the monotone convergence principle uses $\Sigma^0_2$-DNE: $\neg\neg\exists x\in\mathbb{N}\forall y\in\mathbb{N}\varphi_{qf}(x, y) \rightarrow \exists x\in\mathbb{N}\forall y\in\mathbb{N}\varphi_{qf}(x, y)$, but using more induction, weaker $\Sigma^0_1$-LEM($t(x_n)$) suffices (Toftdal 2004).
By the above, $\Sigma^0_1$-LEM in a sense is the weakest amount of classical logic to allow for convergence proofs without computable rate of convergence (Specker sequences).
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Reasoning with $\Sigma^0_1$-LEM

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- **Question:** does forbidding nested repeated use of $\Sigma^0_1$-LEM suffice to guarantee effective fluctuation bounds? **Later: No! But...**
Effective \((B, L)\)-learnability

**Definition (Safarik/K., 2014)**

Consider a \( \Sigma^0_2 \) formula \( \varphi \equiv \exists n \in \mathbb{N} \forall x \in \mathbb{N} \ \varphi_{qf}(x, n, a) \) which is monotone in \( n \), i.e.

\[
\forall n \in \mathbb{N} \ \forall n' \geq n \forall x \in \mathbb{N} \ (\varphi_{qf}(x, n, a) \rightarrow \varphi_{qf}(x, n', a)).
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\(\varphi\) is \((B,L)\)-learnable, if there are function(al)s \(B\) and \(L\) s.t.

\[
\exists i \leq B(a) \forall x \varphi_{qf}(x, c_i, a), \text{ where}
\]

\[
\begin{align*}
c_0 &:= 0, \\
c_{i+1} &:= \begin{cases} 
L(x, a), & \text{for the } x \text{ with } \neg \varphi_{qf}(x, c_i, a) \wedge \forall y < x \varphi_{qf}(y, c_i, a) \text{ if } \exists \\
c_i, & \text{otherwise.}
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\end{align*}
\]
A hierarchy of quantitative forms of Cauchy statements

1. rate $\rho$ of convergence $\Rightarrow$

2. bound ($b := \rho$) on the number of fluctuations $\Rightarrow$

3. $(B, L) -$learnability ($B := b, L(n) := n + 1$) $\Rightarrow$

4. rate of metastability $\Omega$.

Proposition (Safarik/K., 2014)

The hierarchy is strict in the sense that the existence of computable witnesses for level $n$ not even follows from primitive recursive witnesses for level $n - 1$ ($2 \leq n \leq 4$).

Separation of levels 1 and 2: Specker sequences!

The other separations are more complicated (especially 3 versus 4).

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Intuitionistic reasoning and proof mining
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4. **rate of metastability** $\Omega$.

**Proposition (Safarik/K.,2014)**

The **hierarchy is strict** in the sense that the existence of computable witnesses for level $n$ not even follows from primitive recursive witnesses for level $n - 1$ ($2 \leq n \leq 4$).

Separation of levels 1 and 2: Specker sequences!
The other separations are more complicated (especially 3 versus 4).
A metatheorem for \((B, L)\)-bounds

**Theorem** (Safarik/K., 2014)

Let \(\psi_{qf}, \varphi_{qf}\) be quantifier-free s.t. \(\varphi := \exists n \forall x \psi_{qf}(x, n)\) is monotone.

Suppose \(T := \text{HA}^\omega [X, d, \ldots] + \text{AC} + \text{M}^\omega + \text{IP}^\omega\). proves a sentence

\[
\forall a \exists k \in \mathbb{N} \left\{ \left( \forall m \leq k \left( \exists u \in \mathbb{N} \psi_{qf}(u, m, a) \lor \forall v \in \mathbb{N} \neg \psi_{qf}(v, m, a) \right) \right) \rightarrow \exists n \in \mathbb{N} \forall x \in \mathbb{N} \varphi_{qf}(x, n, a) \right\}.
\]

Then one can extract by monotone functional interpretation (self-majorizing) primitive recursive (Gödel) functionals \(B^*, L^*\) s.t. \(\varphi\) is \((B^*, L^*)\)-learnable uniformly in majorants \(a^*\) for \(a\).
The metatheorem is optimal in the sense that the restrictions on the proof do not suffice for effective fluctuation bounds (the separating example for 2./3. formalizes in this context).
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The metatheorem **explains the special form** \((f_2 \circ \tilde{g} \circ f_1)^b(0)\) **of numerous metastability bounds** extracted.
The metatheorem is optimal in the sense that the restrictions on the proof do not suffice for effective fluctuation bounds (the separating example for 2./3. formalizes in this context).

The metatheorem explains the special form \((f_2 \circ \tilde{g} \circ f_1)^b(0)\) of numerous metastability bounds extracted.

If a certain gap condition is satisfied by \((B^*, L^*)\), then one gets fluctuation bounds.
Thank you Professor Troelstra!