THINKING AGAIN ABOUT CHOICE SEQUENCES
for the
Anne Troelstra Memorial Event 2020

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SOME WORKS OF ANNETROELSTRA


1969  Informal Theory of Choice Sequences. Studia Logica XXV


1975  Non-extensional Equality. Fundamenta Mathematicae 82.

MORE WORKS OF ANNE TROELSTRA

1992  Lectures on Linear Logic. (CLSI Lecture Notes 29)
Preserving intellectual inheritance

In particular the idea of

Chore Séquences

Respectful vs. Prioris
Development vs. Ossification

Extreme view

Se vogliamo che tutto rimanga come è, bisogna che tutto cambi.
Brouwer: First Act of Intuitionism

- completely separates mathematics from mathematical language,
- origin in the basic phenomenon of the perception of a more of true,
- the basic operation of mathematical construction is the mental creation of the two-ity of two mathematical systems previously acquired.

It is introspectively realized how this operation continually displaying muddled retention by memory, successively generates each natural number, the infinitely proceeding sequence of the natural numbers, arbitrary finite sequences and infinitely proceeding sequences of mathematical systems, previously acquired....
Brouwer: Second Act of Intuitionism

The second act of intuitionism recognizes the possibility of generating new mathematical entities:

First in the form of infinitely proceeding sequences whose terms are chosen more or less freely from mathematical entities previously acquired; in such a way that the freedom existing perhaps at the first choice may be irreversibly subjected again and again to progressive restrictions at subsequent choices, while all these restricting interventions, as well as the choices themselves, may, at any stage, be made to depend on possible future mathematical experiences of the creating subject.

Secondly in the form of mathematical species...
TROELSTRA'S CAREER

as an exemplary case of

REPECTFUL DEVELOPMENT

Based on

Informal Regime (à la Kreisel)

Axiomatic Method (modern sense)

Informal regime: increasingly delicate analysis of CSs

Axiomatic method: elimination theorems

(a coherent project identified by Joan Moschovakis)
AN EARLY INSIGHT

The gap between

- Spreads \( \sim \) (lawlike) closed sets \( \Pi_1^0 \)
- Analytic sets \( \sim \) images of Borel operations \( \Sigma_1^1 \)

Kreisel's axiom of spread data

\[ F^x A(x) \rightarrow \exists c \in Spd x \in c \land \forall \text{prec } A(y) \]

is "inconsistent" with

Closure under Borel Operations

\[ F^x \forall \beta \exists x \ x = e \mid \beta \]

( Difficulty comes from other constructive principles. )
Consider Browne's operation \( e \) and consider the formula
\[
\forall x \in \text{me} \iff \exists \beta \forall x = e \mid \beta
\]
Spread Data gives
\[
\forall x \in \text{me} \iff \exists e \in \text{Sp} \forall x \in c \in \text{me}
\]
Declare 'generic' \( Y \) and set \( x = e \mid \beta \). Then \( x \in \text{me} \)
and we get
\[
\forall y \in e \in \text{Sp} \forall x \in c \in \text{me}.
\]
(Every analytic set is a union of closed sets!)
APPLY CHOICE PRINCIPLE

AC-CF. \( \forall \alpha \exists a A(\alpha, a) \rightarrow \exists (b_x) \forall \alpha \exists k A(\alpha, b_k) \)

We deduce

\[ \exists (b_x \in \mathbb{SPr}) \forall \alpha \exists k \exists y \in b_k \land b_k \in \mathbb{Lne}. \]

(Every analytic set is a countable union of closed sets \( \sim \Sigma_1 ^0 \equiv \Sigma_2 ^0 \).)

BUT NB (J. Moschovakis) With Church's Thesis,

the true arithmetic hierarchy closes at \( \Sigma_3 ^0 \).
REFINEMENT : BROUWER CONTINUITY

BC-F

\( \forall x \exists a A(x, a) \rightarrow \exists e \exists b_x \forall x A(x, b_e(x)) \)

\( \iff AC-CF + BC-N \)

\( \forall x \exists k A(x, k) \rightarrow \exists e \forall x A(x, e(x)) \).

From this we can ensure that our sequence \((b_n)\) of spreads enumerate exactly those needed:

\( \forall x \exists r \exists e b_n \land b_n \in \text{lme} \),

as well as

\( \forall r \exists k e b_k \land b_k \in \text{lme} \).

So

\( \text{lme} = \bigcup \{ b_x \mid x \in \mathbb{N} \} \).
TROELSTRA'S EXAMPLE I

Take a lawlike \( a : \mathbb{N} \times \mathbb{N} \to 2 \) or \( a : \mathbb{N} \to 2^\mathbb{N} \) or \( \{ a_n \in 2^\mathbb{N} \} \)
a sequence of decidable subsets of \( \mathbb{N} \).

Start by constructing a very simple spread.
Set \( a^+_\alpha(0) = 1 \) \( a^+_\alpha(n+1) = a^+_\alpha(n) \) and define \( S_a \) by

\[
\forall n \in S_a \iff \forall i < |n| \ a^+_\alpha(n_i) = 1
\]

(\text{So the branching is fixed at each level.})

\text{OBSERVE}

\[
\forall \alpha \exists n \ a^+_\alpha(n) = 1 \iff \exists \alpha \in S_a \ . \ \forall i \alpha(i) \neq 0
\]

(\text{\( a^+_\alpha \) inhabited})
TROELSTRA'S EXAMPLE II

(minimal use of Brouwer Operations)

1) For any spread $S$ there is a Brouwer $e_S$ with $S = 1^m e_S$.

2) There is a Brouwer operation $\text{sgn}$ collapsing $\mathbb{N}^\mathbb{N} \rightarrow 2^\mathbb{N}$ in the obvious way.

3) Brouwer operations compose.

So we can set $e = \text{sgn} | e_{S_a}$.

Observe

$$\forall x \exists n \ a_x(n) = 1 \iff \iota = \lambda x . 1 \in 1^m e.$$
Troelstra's Example III

Earlier we saw that for any Brouwer e we can construct a sequence \((b_n)\) of spreads such that

\[ \text{Im e} = \bigcup \{ b_n \mid n \in \mathbb{N} \} \]

So

\[ i \in \text{Im e} \iff \exists n \; i \in b_n \iff \exists n \forall m \; b_n(1^m) = 1 \]

So

\[ \forall n \exists m \; a_n(n) = 1 \iff \exists m \forall n \; b_n(1^m) = 1 \]

Setting \( b_n(m) = b_n(1^m) \) we have shown

\[ \forall a \exists \overline{b} \left( \forall n \exists m \; a(x,n) = 1 \iff \exists x \forall m \; \overline{b}(x,m) = 1 \right) \]

Incompatible with Church's Thesis.
A MODEST PROPOSAL

First act of intuitionism

- Inductive Types (W-Types)

Second act of intuitionism

- Coinductive Types (Haskell)

Brouwer's explanations already indicate mathematical activity combines these.

CS from this perspective?
Brouwer Operations

Given $A, B$ define $K_{AB}$ by

\[
\frac{b \in B}{[b] \in K_{AB}} \quad \frac{s_a \in K_{AB} \quad [a \in A]}{\langle \lambda a.s_a \rangle \in K_{AB}}
\]

~ The initial algebra

\[
K_{AB} \leftarrow \frac{\sim}{B + K_{AB}}
\]

~ Collection of well-founded trees

$A$-branching

\[
\begin{array}{c}
A \\
\vdots \\
\vdots \\
\vdots \\
\text{elements of } B \text{ as leaves}
\end{array}
\]
**APPLICATION**

(INDUCTIVE DEFINTION)

\[
\text{act}: K_A B \times A^N \longrightarrow B \\
\text{act}([b], \alpha) = b \\
\text{act}(\langle \lambda a. S_a \rangle, \alpha) = \text{act}(S_{\alpha, 10}, \alpha')
\]

coming from evident algebra structure

\[
(A^N \Rightarrow B) \leftrightarrow B + (A^N \Rightarrow B)^A
\]

**REMARK** This uses the coalgebra structure

\[
A^N \rightarrow A \times A^N.
\]
We have
\[ K_A B^N \times A^N \rightarrow (K_A B \times A^N)^N \rightarrow B^N \]

Then taking \( A = 1^N \) and using \( B \rightarrow K_B B \rightarrow (K_B B)^N \) we have

\[ K_B B \rightarrow B + (K_B B)^N \rightarrow (K_B B)^N \]

So that \( (K_B B)^N \triangleleft K_B B \) and we have

an action
\[ K_B B \times N^N \rightarrow B^N \]

(As in Kleene Trakletta.)
COMPOSITION OF BROUWER OPERATIONS

We can do this with representations $K_{AB}^N$,

BUT problems with identity associativity

CURE take a retract $L_{AB} \circ K_{AB}^N$

MORAL $K_{AB}^N$ is the wrong coinductive type.

Recall \( K_{AB} = \mu X. B + X^A \).

Define \( L_{AB} = \nu Y. K_A(B \times Y) = \nu Y \mu X (B \times Y + X^A) \).

Have \( L_{AB} \leadsto K_A(B \times L_{AB}) \)

\[ A \leadsto B \times L_{AB} + K_A(B \times L_{AB}) \]

Any element of \( L_{AB} \) is either \([b,s]\) (\( s \in L_{AB} \))
or \( \langle \lambda a, sa \rangle \) (\( sa \in L_{AB} [\lambda a A] \)).

We use this in equations but it gives no universal property.
$\text{APPLICATION (COINDUCTIVE DEFINITION)}$

$\text{act} : L_A B \times A^N \rightarrow B^N$

$\text{act} ([b,s], x) = b, \text{act} (s, x)$

$\text{act} (\langle x_a, s_a \rangle, x) = \text{act} (s_{x_10}, x')$.

Why does this make sense?

- Use $B^N \rightarrow B \times B^N$ a final coalgebra so seek

  $L_A B \times A^N \rightarrow B \times L_A B \times A^N$

- Equivalently

  $K_A (B \times L_A B) \times A^N \rightarrow B \times L_A B \times A^N$
APPLICATION CONTINUED

We tweak our old \( \text{act} : K_A C \times A^N \rightarrow C \) to obtain

\[ \text{act}^* : K_A C \times A^N \rightarrow C \times A^N \]

\[ \text{act}^* ([c], \alpha) = (c, \alpha) \]

\[ \text{act}^* (\langle \lambda, s_2 \rangle, \alpha) = \text{act}^* (s_{\alpha(0)}, \alpha') \]

And this comes from the algebra structure

\[ (A^N \rightarrow C \times A^N) \leftrightarrow C + (A^N \rightarrow C \times A^N)^N \]

**Moral Here?**
DETOUR ON SPREDS

Let $D^+(N)$ be the inhabited decidable subsets of $\mathbb{N}$. (So $D^+N \cong \mathbb{N} \times 2^\mathbb{N}$.)

Coinductive definition

$$Spd \xrightarrow{\sim} \sum_{I \in D^+(N)} Spr^I$$

Traditional coding $\chi : Spd \rightarrow (\text{List}N \rightarrow 2)$

\[
\chi([I, (S \circ i) : i \in I], u) = \begin{cases} 
1 & \text{if } u = \langle \rangle \\
\chi(Su_0, u') & \text{if } u_0 \in I \\
0 & \text{else}
\end{cases}
\]

\]
SPREAD MEMBERSHIP

\[ \chi \in [I, (s_i)] \iff \chi(0) \in I \land \chi' \in S_0(0). \]

What could that mean?

Define conductively:

\[ E \xrightarrow{\sim} \sum_{I \in D^+N} I \times E^I \]

Then have:

\[ E \xrightarrow{\text{for}} N^N \]

from:

\[ E \rightarrow \sum_{I \in D^+N} I \times E^I \rightarrow N \times E \]

and:

\[ E \xrightarrow{\text{and}} \text{Spread} \]

from:

\[ E \rightarrow \sum_{I \in D^+N} I \times E^I \rightarrow \sum_{I \in D^+N} E^I \]

Why are these jointly more?
**Brouwer Category**

- **Objects**: $A, B, C$
- **Maps**: $L_{AB}$
- **Identity**: $\text{id}_A \in L_{AA}$ : $\text{id}_A = \langle \lambda a. [a, \text{id}_A] \rangle$
- **Composition**: $L_{BC} \times L_{AB} \rightarrow L_{AC}$

\[
[c, t] \circ s = [c, t \circ s]
\]
\[
\langle \lambda b, t b \rangle \circ [b, s] = t b \circ s
\]
\[
\langle \rightarrow \circ \lambda a \rangle s = \lambda a \langle \rightarrow \circ s a \rangle
\]
BROUWER TOPOLOGY

For each $A$ canonical covers are determined by the elements of $K_A$.

Each such determines a set of nodes in List $A$ and each node $u$ determines an idempotent $e_u$

$$e_x = id \quad e_{u_0 \cdot u_1} = \langle x_0, [u_0, e_{u_1}] \rangle$$

Cover is given by the maps factoring through these.
YET ANOTHER CS MODEL

Use the monoid \( L^N \) with the induced topology in the style of Ponsman van der Hoeven - Moerdijk

MAIN FEATURE

The representable does not give the internal \( I^{N^N} \) of \( \text{vdH-M} \) on spread data. But this time it is "big" and extensionality fails.
1975
Non-extensional equality

1996
Choice Sequences: a retrospect

From 1996
I follow another analytic approach which highlights the "intensional" aspects of choice sequences.

My last meeting with Anne Troelstra at the Utrecht Topology Feest 2018 (FvIM)

Our discussion:
- intensional aspects of CS
- history of botany