The complexity of mathematical practice

- *Proofs and Refutations*: dealing with proofs is a complex matter

- Is Euler's theorem an exemplar or not?

- What to do when there is no proof?

- Why prove one case of a universal statement? (cfr. Fermat)

- What was Hilbert doing in 1900 in Paris?

- Sketch of a theory of mathematical practice:
  - macroscopic level
  - mesoscopic level
  - microscopic level

- Philosophical impact of such a theory
  - explanatory value of proofs
  - role of aesthetics in quality of proofs
 - what is an "interesting" problem?

The macroscopic level

- Discussion still dominated by the question of revolutions in mathematics
- What are they, if anything:
  - ontological?
  - epistemological?
  - semiotical?
- Philosophical importance obvious

What does seem reasonable is that large-scale periods can be identified (Koetsier):

- The Pythagorean "tradition"
- The Euclidean "tradition"
- The "Age of Rigour"
- The foundationalist period
- …

Fundamental problem: how to characterize such a period?
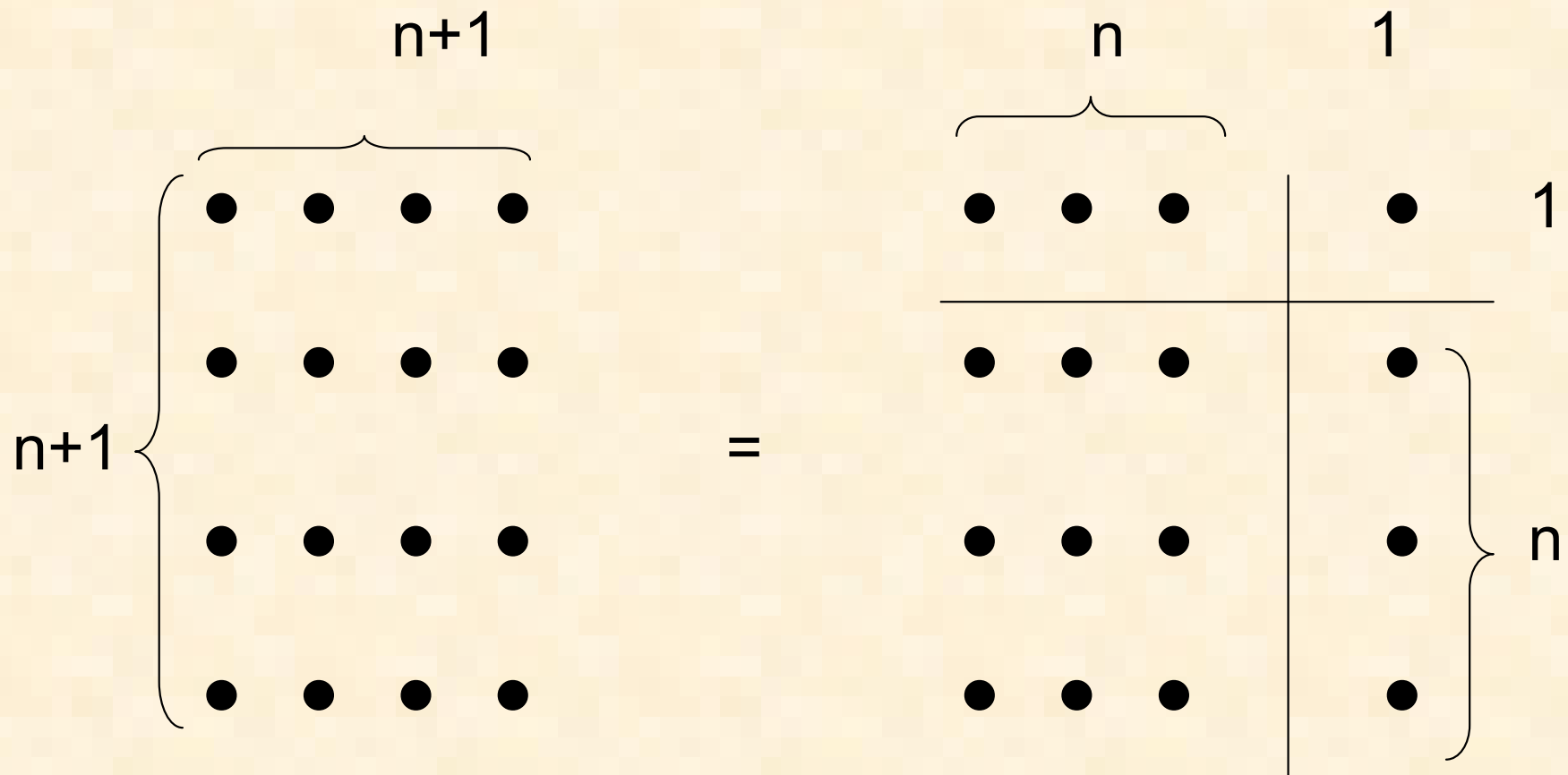
The mesoscopic level

- Level of "research programmes"

- *Examples*:

(a) *The Erlanger Program* (Felix Klein)

'In geometry, Felix Klein proposed that the many varieties of space provided by non-Euclidean and other geometries could be classified and hence organized in terms of their groups of symmetries - the full linear group, the orthogonal group, the projective group, and others.' (MacLane, 1986: 407)

(b) The *Langlands Program*

- Infinite dimensional representations of Lie groups as a tool to solve problems in number theory

'... Langlands' program is a synthesis of several important themes in classical number theory. It is also - and more significantly - a program for future research. This program emerged around 1967 in the form of a series of conjectures, and it has subsequently influenced recent research in number theory in much the same way the conjectures of A. Weil shapes the course of algebraic geometry since 1948.' (Gelbart, 1984: 178)

(c) *Hilbert's Program*: no comment

- Interesting detail: in the notebooks (notebook Cod. ms. Hilbert 600:3) a 24th problem is mentioned:
- Find algorithms that, given a proof of a theorem, reduce the proof to its shortest and/or simplest version
- *Conjecture*: failure to formulate the problem in a formally satisfactory way

(Larry WOS & Ruediger THIELE, "Hilbert's New Problem", *Bulletin of the Section of Logic*, vol. 30, nr. 3, 2001, pp. 165-175.)

(d) The *classification of finite simple groups*:

'The state of the original proof is such that if everyone who worked on it should vanish, it would be very hard for future generations of mathematicians to reconstruct the proof out of the literature.' (Cipra, 1996: 89)

- 3 aims:

  (i) To make uniform the different proof methods that have been used over the thirty year period

  (ii) To reduce the size of the proof to something like 5.000 pages, perhaps even shorter and to publish the proof as a *single* proof

  (iii) To eliminate all errors present

(e) *Category Theory Program* (?)

'The situation bears some resemblance to that in geometry after the discovery of consistency proofs for non-Euclidean geometry showed that there was not one geometry, but many. This meant that geometries could be formulated with many different systems of axioms, some of which were relevant to higher analysis and some to physics. ... Similarly, the initial idea of a collection leads to substantially different versions of set theory, some of which ... have relevance to other parts of Mathematics, though not yet (?) to Physics.' (MacLane, 1986: 385-386)

Elements that characterize a research program

- Background theory/theories
- Conjectures
- Estimates of likelihood to find answers (positive or negative)
- (Partial) agreement on methods to use
- "Paradigmatic" examples

The daily life of a mathematician

- Proof methods
  - The reductio proof
  - Proof by infinite descent
  - Proof by cases
  - Mathematical induction
  - Career induction
  - (Partial) computer proof
  - "Experimental" proof

- *Note*: different from logician's idea of proof

A. R. D. MATHIAS

A TERM OF LENGTH 4 523 659 424 929

ABSTRACT. Bourbaki suggest that their definition of the number 1 runs to some tens of thousands of symbols. We show that that is a considerable under-estimate, the true number of symbols being that in the title, not counting 1 179 618 517 981 links between symbols that are needed to disambiguate the whole expression.

- Heuristic methods related to proofs:
  Most famous example: $\Sigma\ 1/n^2 = \pi^2/6$

Euler first reasons about polynomials of finite even degree 2n, of the following form:

$$b_o - b_1 x^2 + b_2 x^4 - \ldots + (-1)^n b_n x^{2n} = 0$$

with roots: $r_1, -r_1, r_2, -r_2, \ldots, r_n, -r_n$.

He shows that the following holds:

$$b_1 = b_o(1/r_1^2 + 1/r_2^2 + \ldots + 1/r_n^2) \qquad\qquad (*)$$

Now take the polynomial of infinite degree:

$$1 - x^2/3! + x^4/5! - x^6/7! + \ldots = 0,$$

with roots: $\pi, -\pi, 2\pi, -2\pi, 3\pi, -3\pi, \ldots$,

(as it is the series expansion of sin(x)/x), will satisfy (*), thus

$$1/3! = 1/\pi^2 + 1/4\pi^2 + 1/9\pi^2 + \ldots,$$

or: $1 + 1/4 + 1/9 + \ldots = \pi^2/6.$ \qquad QED (?)

- Perhaps less known:

$\Sigma\ 1/(2n)^2 = \pi^2/24$, i.e., the sum of the reciprocals of all even squares,

$\Sigma\ 1/(2n+1)^2 = \pi^2/8$, i.e., the sum of the reciprocals of all odd squares,

$\Sigma\ 1/n^4 = \pi^4/90$, i.e., the sum of the reciprocals of all fourth powers.

- A fine illustration of the mathematicians' idea of what a proof should look like is *the Apéry case*

$$Z(s) = \Sigma_n \ 1/n^s$$

for $n = 2k$, $Z(2k) = (-1)^{k-1}(2\pi)^{2k}B_{2k}/(2.(2k)!)$, where B_{2k} is the 2k-th Bernoulli number, i.e., the 2k-th coefficient in the equation: $x/(e^x-1) = \Sigma \ B_i x^i/k!$

Are Z(3), Z(5), ..., rational or irrational? In June 1978, Roger Apéry presented a proof of the irrationality of Z(3).

What was "wrong" with Apéry's formally correct proof?

Mathematicians gave the following comments:

(i) The proof was 'mysterious' and consisted of a series of 'miracles'.

$$n^3 u_n = (34n^3 - 51n^2 + 27n - 5)u_{n-1} - (n-1)^3 u_{n-2}.$$

Apéry claimed the following: if one starts with $u_o = 1$ and $u_1 = 5$, then all u_n are integers!

(ii) The proof offers no clues at all for other values of Z(s) for s = 2n+1.

(iii) Apéry did not use any 'new' proof methods.

Short aside:

- Source of mathematical beauty can be found here:

Adaptation of Birkhoff's formula

$$M = O/C$$

Where   O = importance of the theorem

                  C = complexity of the proof

Thus: a simple proof for an important statement is the best one can have

- Finding the "right" concepts:

Nice example: Bernhard Riemann (1859):

"*Uber die Anzahl der Primzahlen unter einer gegebenen Grösse*" ("*On the Number of Prime Numbers less than a Given Quantity*")

From
$$\Gamma^*(n) = \int_0^\infty e^{-x} x^n \, dx$$

he defines:

$$\zeta(s) = \Gamma^*(-s).2\pi^{s-1}.2\sin\frac{s\pi}{2}.\zeta(1-s)$$

And then he defines:

$$\xi(s) = \Gamma^*(\frac{s}{2}).(s-1).\Gamma^{*-\frac{s}{2}}.\zeta(s)$$

And, in addition, a new function is defined:

$$J(x) = \frac{1}{2}\left[\sum_{p^n < x}\frac{1}{n} + \sum_{p^n \leq x}\frac{1}{n}\right]$$

- In short: the paper is almost entirely about generating new concepts (and the famous hypothesis is really a side remark)

"One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation."
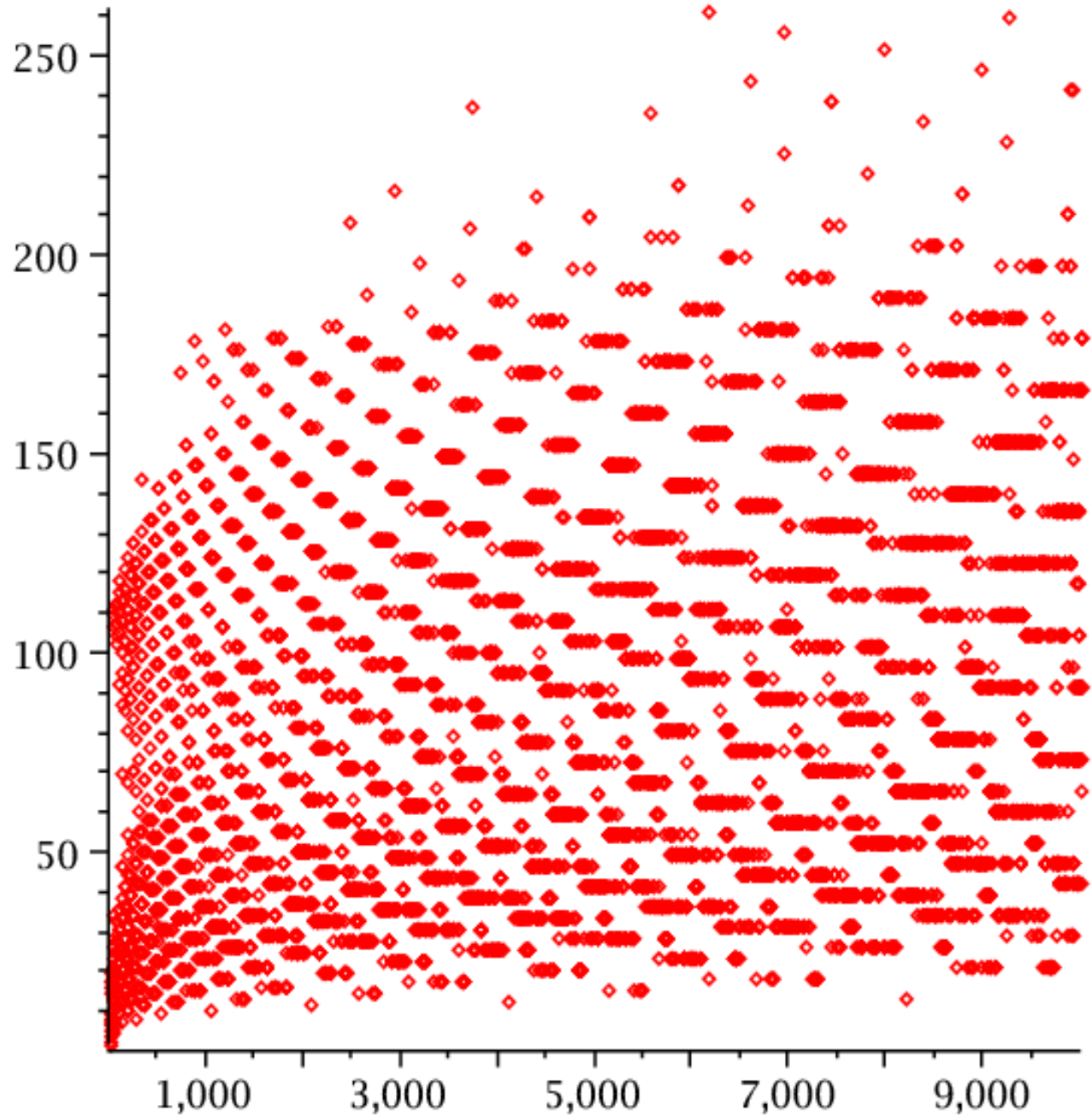
- "Experimental" mathematics to gain insights or to find patterns

- Example: Collatz Conjecture

Given a natural number n, repeat the following operation T:

either $\quad n/2 \quad$ if n is even

or $\quad (3n+1)/2 \quad$ if n is odd

Take $7 - 11 - 17 - 26 - 13 - 20 - 10 - 5 - 8 - 4 - 2 - 1$

Conjecture: for every n, 1 will occur in the sequence

Ranging from 1 to 9999, this graph shows the "total stopping time", i.e., the length of the sequence from beginning to the first time 1 is encountered

- This list is not exhaustive
  - Use of diagrams in proofs
  - The general issue of visualisation
  - The matter of style
  - Use of metamathematical considerations (what if theproblem is undecidable?)

On the microscopical level the mathematician's activity is a (rather) complex mixture of different practices, just one of which is theorem proving

The curious nature of the mathematical universe

- Why only partial success is to be expected
- Example 1:

Let $\pi(n)$ = number of prime numbers $\leq n$.
Let Li(n) be the function:

$$\int_2^n (1/\ln(x)\,dx$$

where ln(x) is the natural logarithm of x.

For finite values for n, one notes, by direct calculation, that, although the difference is small,
$$Li(n) > \pi(n)$$

Calculations up to 10^9 showed that this is the case. It seemed more than reasonable to conclude that this is always the case.

Littlewood has shown that the difference

$Li(n) - \pi(n)$ changes sign *infinitely many* times!

The first estimate for what value of n this is supposed to happen, was given by Skewes:

$10^{\wedge}(10^{\wedge}(10^{\wedge}34)))$

- Example 2:

$$\pi = [(1/10^5)[\Sigma_n \, e^{-n^2/10\wedge 10}]]^2,$$

where n goes from $-\infty$ to ∞

This formula gives the correct value for π up to 42 billion digits and only then do things go wrong!

- In addition: the road to the universe is just as tricky: *proofs from the unexpected*

Example:

Consider a real function f: R → R.

A real function f is *symmetric* if f(x) = f(-x),

anti-symmetric if f(x) = -f(-x)

Show that any real function is the sum of a symmetric and anti-symmetric function.

Probably one would tend to 'subtract' from f a symmetric function g and then try to show that f - g is an anti-symmetric function under certain conditions. Whereas the answer is just this:

$$f(x) = [f(x) + f(-x)]/2 + [f(x) - f(-x)]/2.$$

Obviously $f(x) + f(-x) = f(-x) + f(x)$ and $f(x) - f(-x) = -[f(-x) - f(x)]$ QED

To conclude:

- Philosophers of mathematics cannot restrict themselves to mere foundational problems, e.g., what are numbers, …

- "Stable" notions (proof, axiom, …) are to be seen rather as "flexible" notions

- The additional elements have philosophical importance: concept generation, justification, explanation, style, beauty