Logic As A Tool For Building Theories

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Introduction

• Overview
• The Plan

Case Study I: Modal and Temporal Logic

Case Study II: \(\lambda\)-calculus

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Basic Concepts

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Practice-based Philosophy of Logic?
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What is the practice?
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**What is the practice?**

Huge effect of Computer Science on Logic over the past 5 decades:

- new ways of *using* logic
- new attitudes to logic
- new questions
- new methods
Practice-based Philosophy of Logic?

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Hence new perspective on the question:

What logic is — and should be!
Before (and while) trying to extract general points, some case studies:
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- modal and temporal logic: verification and model-checking
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- modal and temporal logic: verification and model-checking
- λ-calculus
- coalgebra
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Aims: not to put forward any philosophical theses, but to provide some materials and raise some questions.
Case Study I: Modal and Temporal Logic

Introduction

Case Study I: Modal and Temporal Logic
• Changing Perspectives
• Some Issues
• Some Remarks and Questions for P-B PoL
• The 'Next 700' Problem

Case Study II: $\lambda$-calculus
Case Study III: Category Theory and Coalgebra

Basic Concepts
Final Remarks
• From ‘philosophical logic’ to computer-assisted verification. From metaphysics to (not just potentially but actually) applied mathematics.
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• More concretely: from ‘possible worlds’ and ‘accessibility’ to states and transitions. Systems of states evolving under discrete transitions turn up in a huge variety of situations. (Communications protocols, hardware circuits, software, nowadays biological and physical systems . . . ). Modal and temporal logics are canonical formalisms for expressing and reasoning about properties of such systems.
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A transition system:

![Transition System Diagram](image-url)
Some Issues
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  - **model-checking**. Given a system description (a transition system) $S$, and a property $\phi$ we wish the system to satisfy, check if $S \models \phi$. This has become an enormously influential paradigm over the past 25 years. Much of the real value lies in cases where the property is **not** satisfied, and we get a trace which can lead us to the bug.
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• Huge expansion to cover real-time, probabilistic and hybrid systems, and of applications to include biological systems, security, networks, agent-based modelling, control systems etc.
Some Remarks and Questions for P-B PoL

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- There are passionate methodological debates within this applied field, e.g. ‘linear time vs. branching time’, which are fertile ground for conceptual and philosophical analysis. Feasibility becomes a major new criterion, and approximate answers must be considered. Such issues are already deeply embedded in physics, but rarely studied philosophically — they should be!
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- A deep conceptual issue of logic in CS: the ‘next 700 . . . problem’. 
The ‘Next 700’ Problem
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A profusion of possibilities, in e.g.

- programming languages
- type systems
- process calculi
- behavioural equivalences
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Cf. André Weil: he compared finding the right definitions in algebraic number theory — which was like carving adamantine rock — to making definitions in the theory of uniform spaces (which he founded), which was like sculpting with snow.
Case Study II: $\lambda$-calculus

- The $\lambda$-calculus
- Remarks
- Types and Curry-Howard Correspondence
- Developments
- Domain Theory
- Some Remarks and Questions for P-B PoL

Case Study III:
Category Theory and Coalgebra

Basic Concepts
Final Remarks
The $\lambda$-calculus

$\lambda$-calculus: a pure calculus of functions.
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Variables $x, y, z, \ldots$

Terms

\[
t ::= x \mid tu \mid \lambda x. t
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application abstraction
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application \hspace{2cm} abstraction

The basic equation governing this calculus is $\beta$-conversion:

$$(\lambda x. t)u = t[u/x]$$

E.g.

$$(\lambda f. \lambda x. f(fx))(\lambda x. x + 1)0 = \cdots 2.$$
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By orienting this equation, we get a ‘dynamics’ — \( \beta \)-reduction

\[
(\lambda x. t)u \rightarrow t[u/x]
\]
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Hence also $\Omega \equiv \omega \omega$, which **diverges**:

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Also, $Y \equiv \lambda f. (\lambda x. f (x x))(\lambda x. f (x x))$ — recursion.

$$Y t \rightarrow (\lambda x. t (x x))(\lambda x. t (x x)) \rightarrow t((\lambda x. t (x x))(\lambda x. t (x x))) = t(Y t).$$
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  $Yt \rightarrow (\lambda x. t(xx))((\lambda x. t(xx))) \rightarrow t((\lambda x. t(xx))((\lambda x. t(xx)))) = t(Yt)$.

Historically, Curry extracted $Y$ from an analysis of **Russell’s Paradox**.

**Remarks**
Simple Type System for $\times$, $\to$.

Variable

$$\Gamma, x : t \vdash x : T$$

Product

$$\frac{\Gamma \vdash t : T \quad \Gamma \vdash u : U}{\Gamma \vdash \langle t, u \rangle : T \times U} \quad \frac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_1 v : T} \quad \frac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_2 v : U}$$

Function

$$\frac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x. t : U \to T} \quad \frac{\Gamma \vdash t : U \to T \quad \Gamma \vdash u : U}{\Gamma \vdash tu : T}$$
**Natural Deduction System for \( \land, \supset \)**

**Identity**

\[
\Gamma, A \vdash A \quad \text{Id}
\]

**Conjunction**

\[
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B} \quad \land\text{-intro}
\]

\[
\frac{\Gamma \vdash A \land B}{\Gamma \vdash A} \quad \land\text{-elim-1}
\]

\[
\frac{\Gamma \vdash A \land B}{\Gamma \vdash B} \quad \land\text{-elim-2}
\]

**Implication**

\[
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \quad \supset\text{-intro}
\]

\[
\frac{\Gamma \vdash A \supset B}{\Gamma \vdash A} \quad \supset\text{-elim}
\]
If we equate

\(\land \equiv \times\)

\(\supset \equiv \to\)

they are the same!
Developments

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  The λ-calculus and combinatory logic were a neglected corner of logic studied by a handful of people until Computer Science — initially Strachey and Landin, with a part played by Roger Penrose — put it centre stage in logical methods in CS.
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These calculi in turn put the study of substitution centre-stage — not such a humble topic! Russell’s paradox, cut-elimination, linearity and resources, decidability, complexity, . . .
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Paradoxes: not just biting the bullet — not bugs but features! Recursion, fixpoints, the creative uses of computationally specified infinite objects.
Providing extensional models for \( \lambda \)-calculus — spaces satisfying

\[
D \cong [D \rightarrow D]
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led Dana Scott to **Domain Theory**.
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A discussion of domain theory emphasizing conceptual aspects in my article in the Handbook of Philosophy of Information (ed. van Benthem and Adriaans, Elsevier 2008).
Some Remarks and Questions for P-B PoL

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- Game Semantics, full abstraction and full completeness. Again, see my article in the Handbook of Philosophy of Information (and, hopefully, forthcoming article in SEP).
Case Study III: Category Theory and Coalgebra

Some Theses About Category Theory
Coalgebras

Basic Concepts

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  Examples: Cohomology, categorification, the microcosm principle.
Category theory allows us to **dualize** our entire discussion of algebras to obtain a notion of **coalgebras of an endofunctor**. However, while algebras abstract a familiar set of notions (inductive data types, structural recursion), colagebras open up a new and rather unexpected territory, and provides an effective abstraction and mathematical theory for a central class of computational phenomena:

- Programming over **infinite data structures**: streams, lazy lists, infinite trees . . .
- A novel notion of **coinduction**
- Modelling **state-based computations** of all kinds
- The key notion of **bisimulation equivalence** between processes.
Basic Concepts

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Basic Concepts
- \(F\)-Coalgebras
- Final \(F\)-coalgebras
- Labelled Transition Systems
- Transition Graphs as Coalgebras
- The Final Coalgebra
- Some Remarks and Questions for P-B PoL

Final Remarks
Let $F : C \rightarrow C$ be a functor.

An $F$-coalgebra is an arrow $\gamma : A \rightarrow FA$ for some object $A$ of $C$. We say that $A$ is the carrier of the coalgebra, while $\gamma$ is the behaviour map.

An $F$-coalgebra homomorphism from $\gamma : A \rightarrow FA$ to $\delta : B \rightarrow FB$ is an arrow $h : A \rightarrow B$ such that

\[
\begin{array}{ccc}
A & \xrightarrow{\gamma} & FA \\
\downarrow{h} & & \downarrow{Fh} \\
B & \xrightarrow{\delta} & FB
\end{array}
\]

$F$-coalgebras and their homomorphisms form a category $F-Coalg$. 
An $F$-coalgebra $\gamma$ is **final** if for every $F$-coalgebra $\delta$ there is a unique homomorphism from $\delta$ to $\gamma$.

**Proposition 1**  
If a final $F$-coalgebra exists, it is unique up to isomorphism.

**Proposition 2 (Lambek Lemma)**  
If $\gamma : A \rightarrow FA$ is final, it is an isomorphism.
Let \( A \) be a set of \textbf{actions}. A \textit{labelled transition system over} \( A \) is a coalgebra for the functor

\[
\text{LT}_A : \text{Set} \longrightarrow \text{Set} :: X \mapsto \mathcal{P}_f(A \times X).
\]

Such a coalgebra

\[
\gamma : S \longrightarrow \mathcal{P}_f(A \times S)
\]

can be understood operationally as follows:

- \( S \) is a set of \textbf{states}
- For each state \( s \in S \), \( \gamma(s) \) specifies the possible \textbf{transitions} from that state. We write \( s \xrightarrow{a} s' \) if \( (a, s') \in \gamma(s) \). We think of such a transition as consisting of the system performing the action \( a \), and changing state from \( s \) to \( s' \). Note that we regard actions as directly \textbf{observable}, while states are not.
Note that any labelled transition graph (or “state machine”) with labels in \( A \) is a coalgebra for \( \text{LT}_A \).

**Examples 1.**

This corresponds to the coalgebra \((\{1, 2\}, \gamma)\)

\[
\begin{align*}
\gamma : 1 &\mapsto \{(a, 1), (b, 2)\}, \\
2 &\mapsto \{(c, 2)\}
\end{align*}
\]

2.

\[
\begin{align*}
1 &\mapsto \{(b, 2), (c, 1)\}, \\
2 &\mapsto \{(a, 1), (a, 3)\}, \\
3 &\mapsto \emptyset
\end{align*}
\]
The key point is this.

**Proposition 3**  *For any set $A$ of actions, there is a final $L T_A$-coalgebra $(Proc_A, \text{out})$.*
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We think of elements of the final coalgebra as *processes*. The final coalgebra provides a “universal semantics” for transition systems, which is “fully abstract” with respect to observable behaviour.
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All of this generalizes to a large class of endofunctors.
Some Remarks and Questions for P-B PoL

- Coalgebras naturally model state-based systems. They provide a promising basis for reconciling *ontic* and *epistemic* views of states. The final coalgebra is a universal solution — hence unique up to isomorphism — to the problem of constructing states as determined purely by their observational behaviour.
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• Corecursion, coinduction: mathematically well-founded treatment of non-well-founded objects.
Coalgebras naturally model state-based systems. They provide a promising basis for reconciling **ontic** and **epistemic** views of states. The final coalgebra is a universal solution — hence unique up to isomorphism — to the problem of constructing states as determined purely by their observational behaviour.

Coalgebraic logic. A generalized modal logic which can be **read off systematically** from the type functor \( T \). Generalized duality theory.

Corecursion, coinduction: mathematically well-founded treatment of non-well-founded objects. Examples: non-well-founded sets, even non-well-founded proofs!
Final Remarks

- Logic As A Tool For Building Theories
- Some Challenges for Practice-Based Philosophy of Logic
Computer Science theories of:

- Processes of various kinds, how to mathematically describe and reason about them.
- Information: statics of information representation, dynamics of information flow.
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Logic in the mode of open-ended, outward-reaching modelling, rather than conservative codification.
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Considerable potential beyond Computer Science: in physics, biology, cognitive and social sciences etc.
Analyze a real, contemporary research programme in mathematics, logic or theoretical computer science.
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Study the choices made, the reasons given, the methodological disagreements, what these were really about, why certain contributions were decisive, why conceptual arguments about approaches were decided in a certain way.
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Contrast: Philosophy of Physics vs. Philosophy of Logic and Mathematics.