Paradoxes of Interacting Modalities

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Joint work with Johannes Stern!

Content

Modalities

Modalities can be treated as:

- modal operators, $\Box\varphi$, or
- modal predicates, $N(\ulcorner\varphi\urcorner)$.

Early proponents of the syntactical treatment:

- Carnap 1934: The logical syntax of language.
- Quine 1953: Three grades of modal involvement.

Expressive power

Some advantages of the syntactical approach:

- expressive power.
- symmetry to truth.
- good for interaction.

Examples:

- $\forall x \forall y (N(x) \wedge N(x \underset{\cdot}{\rightarrow} y) \rightarrow N(y))$.
- $\Box\varphi \wedge \Box(\varphi \rightarrow \psi) \rightarrow \Box\psi$.
- $\forall x (T(x) \rightarrow P(\underset{\cdot}{K}\dot{x}))$.
- $\exists x (T(x) \wedge \neg P(\underset{\cdot}{K}\dot{x}))$.

Diagonallemma

- In the setting of syntactical treatments we usually assume some theory of syntax in the background.
- For this we can assume some theory of arithmetic sufficiently strong, *PA*.

Lemma (Diagonallemma)

Let T be a theory containing Q. Let ϕ be a formula with one free variable, then there is a sentence δ such that

$$T \vdash \delta \leftrightarrow \phi(\ulcorner \delta \urcorner)$$

δ is a fixed-point.

Modal Paradoxes

Theorem (Montague)

Let $\mathcal{L}_N$ be the arithmetical language extended by the predicate symbol 'N', Σ a theory extending Q. If for every closed formula ϕ of $\mathcal{L}_N$,

$$\text{(T)} \quad \Sigma \vdash N\ulcorner\phi\urcorner \rightarrow \phi$$

$$\text{(Nec)} \quad \Sigma \vdash \phi \Rightarrow \Sigma \vdash N\ulcorner\phi\urcorner$$

then Σ is inconsistent.

The proof uses the fixed-point lemma and specifically makes use of the fixed-point

$$\delta \leftrightarrow \neg N\ulcorner\delta\urcorner$$

> *Thus if necessity is to be treated syntactically, that is, as a predicate of sentences, as Carnap and Quine have urged, then virtually all of modal logic [...] must be sacrificed. (Montague 1963, p.294)*

Mistakes

Here is what I consider one of the biggest mistakes of all in modal logic: concentration on a system with just one modal operator. The only way to have any philosophically significant results in deontic logic or epistemic logic is to combine these operators with: Tense operators (otherwise how can you formulate principles of change?); the logical operators (otherwise how can you compare the relative with the absolute?); the operators like historical or physical necessity (otherwise how can you relate the agent to his environment?); and so on and so on. (Scott 1970, 143)

Multiple Modal Predicates

Question: What happens in the case of two or more modal predicates?

- Base language = language of arithmetic $\mathcal{L}_A$.
- Language expansions by one place predicates $N_1, N_2, \ldots$.
- First-order axiomatic theories $\Sigma_{N_1}, \Sigma_{N_2}, \ldots$ containing PA.
- Axioms of interaction:
  - Example: $\forall x(Tx \rightarrow PK\dot{x})$ (Knowability Principle).
  - Axiom schemata expanded to formulas containing multiple modal predicates.

Questions of interest

- If we do not allow for axioms of interaction we should not be able to derive new theorems. $\Sigma_{N_1} \cup \Sigma_{N_2}$ should be conservative over Σ_{N_1}.
- If we allow for axioms of interaction the proof-theoretic strength may be significantly increased. Example Halbach's modalized disquotationalism: Two theories $\Sigma_{N_1}, \Sigma_{N_2}$ which are conservative extensions of *PA* joined together with some interaction axioms give a theory significantly stronger than *PA*.
- Interaction axioms may even lead to **new** paradoxes!

Fitch

Theorem (Halbach 2008)

Let $\mathcal{L}_{NK}$ be the arithmetical language extended by the predicate symbols 'N' and 'K', Σ a theory extending Q. If for every closed formula ϕ of $\mathcal{L}_{NK}$;

$$(T_K)\ \Sigma \vdash K\ulcorner\phi\urcorner \rightarrow \phi$$

$$(Nec_N)\ \Sigma \vdash \phi \Rightarrow \Sigma \vdash N\ulcorner\phi\urcorner$$

$$(F)\ \Sigma \vdash \phi \rightarrow \neg N\ulcorner\neg K\ulcorner\phi\urcorner\urcorner$$

then Σ is inconsistent.

The proof uses the fixed-point lemma and specifically makes use of the fixed-point

$$\delta \leftrightarrow N\ulcorner\neg K\ulcorner\delta\urcorner\urcorner$$

Fitch Proof

Proof.

1	$\delta \leftrightarrow N\ulcorner\neg K\ulcorner\delta\urcorner\urcorner$	Fixed-point lemma
2	$\delta \rightarrow \neg N\ulcorner\neg K\ulcorner\delta\urcorner\urcorner$	F
3	$\neg\delta$	1,2
4	$\neg\delta \rightarrow \neg K\ulcorner\delta\urcorner$	T_K
5	$\neg K\ulcorner\delta\urcorner$	3,4
6	$N\ulcorner\neg K\ulcorner\delta\urcorner\urcorner$	5, Nec_N
7	$\neg N\ulcorner\neg K\ulcorner\delta\urcorner\urcorner$	1,3

□

No Future

Theorem (Horsten/Leitgeb 2001)

Let $\mathcal{L}_{H,G}$ be the arithmetical language extended by the predicate symbols 'H' and 'G', Σ a theory extending Q. If for every closed formula ϕ, ψ of $\mathcal{L}_{H,G}$,

$$
\begin{aligned}
(Nec_G)\ & \Sigma \vdash \phi \Rightarrow \Sigma \vdash G\ulcorner\phi\urcorner \\
(Nec_H)\ & \Sigma \vdash \phi \Rightarrow \Sigma \vdash H\ulcorner\phi\urcorner \\
(K_G)\ & \Sigma \vdash G\ulcorner\phi \to \psi\urcorner \to (G\ulcorner\phi\urcorner \to G\ulcorner\psi\urcorner) \\
(K_H)\ & \Sigma \vdash H\ulcorner\phi \to \psi\urcorner \to (H\ulcorner\phi\urcorner \to H\ulcorner\psi\urcorner) \\
(Con_H)\ & \Sigma \vdash H\ulcorner\neg\phi\urcorner \to \neg H\ulcorner\phi\urcorner \\
(HF)\ & \Sigma \vdash \phi \to H\ulcorner\neg G\ulcorner\neg\phi\urcorner\urcorner
\end{aligned}
$$

then Σ is inconsistent.

The proof makes use of the following fixed-point:

$$\delta \leftrightarrow H\ulcorner\neg G\ulcorner\delta\urcorner\urcorner$$

Extended Montague

Theorem

Let $\mathcal{L}_{NK}$ be the arithmetical language extended by the predicate symbols 'N' and 'K', Σ a theory extending Q. If for every closed formula ϕ of $\mathcal{L}_{NK}$;

$$(Nec_K)\ \Sigma \vdash \phi \Rightarrow \Sigma \vdash K\ulcorner\phi\urcorner$$

$$(Nec_N)\ \Sigma \vdash \phi \Rightarrow \Sigma \vdash N\ulcorner\phi\urcorner$$

$$(T_{NK})\ \Sigma \vdash N\ulcorner K\ulcorner\phi\urcorner\urcorner \rightarrow \phi$$

then Σ is inconsistent.

The proof uses the fixed-point:

$$\delta \leftrightarrow \neg N\ulcorner K\ulcorner\delta\urcorner\urcorner$$

Question of reducibility

I want to address the following questions:

- What fixed points are relevant for the derivation of the inconsistencies?
- Are these genuinely new paradoxes of interaction?
- Or, do they just arise because the underlying modal logic for a single operator was strengthened?
- I.e., can they be reduced to the paradoxes of the single modal predicates?
- What kind of reducibility is appropriate in this context?

Problem of Reducibility

Problem

We are dealing with inconsistent theories.

- Classical concepts of reductions seem inapplicable.
- Consider for example relative interpretations.
  Any theory is interpretable in an inconsistent theory.
- Find a different approach!

Explicating Reducibility

Idea

Translate the question to a framework that allows:

- that there are still inconsistencies in cases of interaction, and
- for two possibilities in cases without interaction:
  - consistency $\Rightarrow$ nonreducibility.
  - inconsistency $\Rightarrow$ reducibility.

$\Rightarrow$ Modal Operator Logics with fixed-points or Diagonalized Modal Logic.

Diagonalizing Modal Logic

The idea of Diagonalized Modal Logic, DML, goes back to Smorińsky's DOL (1985).

- The paradoxes in the case of modal predicates arise from instances of the diagonal lemma.
- The usual operator setting does not contain fixed points.
- It is possible to extend the logics by adding new fixed point constants and fixed point axioms.

For formulas $\psi(q)$ we add a new fixed point constant $\delta_{\psi(q)}$ to the language and also add the fixed-point axiom

$$\delta_{\psi(q)} \leftrightarrow \psi(\delta_{\psi(q)})$$

Remark: to avoid trivialization q has to be guarded in ϕ, i.e. every occurrence of q in ϕ is in the scope of at least one $\Box$. This is also in line with the fact that in the arithmetical case the fixed point occurs in $\ulcorner\urcorner$.

Diagonalizing Multi Modal Logic

For multi modal logics we have two possibilities of adding fixed points:

- fixed point constants for each modal operator, or
- fixed points for both modal operators.

We can then consider whether

- the inconsistency only arises in the second case $\Rightarrow$ nonreducibility, or
- also in the first $\Rightarrow$ reducibility.

Diagonalized Modal Logic (DML)—Part 1

Definition (DML-language)

Let $\mathcal{L}_\Box$ be a standard modal operator language containing a modal operator $\Box$. Set $L_0 =_{DEF} \mathcal{L}_\Box$ and define

$$L_{n+1} =_{DEF} L_n \cup \{\delta_{\psi(q)} \mid \psi(q) \in L_n \ \& \ q \text{ is guarded in } \psi\}$$

Finally, set the language of DML, $\mathcal{L}^F_\Box$, to be the union of all L_n, i.e.

$$\mathcal{L}^F_\Box =_{DEF} \bigcup_{n \in \omega} L_n$$

Diagonalized Modal Logic (DML)—Part 2

Definition (DML-logic)

Let $\mathcal{S}$ be an arbitrary modal logic axiomatized by axioms $X_0, \ldots, X_{r-1}$ and rules $R_0, \ldots, R_{l-1}$ formulated in $\mathcal{L}^F_\Box$, then $\mathcal{S}^F$ is the diagonal extension of $\mathcal{S}$, i.e. the closure of the set

$$\mathcal{S} \cup \{\psi(\delta_{\psi(q)}) \leftrightarrow \delta_{\psi(q)} \mid \psi \in \mathcal{L}^F_\Box \ \& \ q \text{ is guarded in } \psi\}$$

under modus ponens and $R_0, R_1, \ldots$ and R_{l-1}

Diagonalized Modal Logic (DML)—Example

Example

Let $\mathcal{S}$ be the modal logic axiomatized by (T),(K) and (Nec), then $\mathcal{S}^F$ is this logic augmented by the axioms:

$$\psi(\delta_{\psi(q)}) \leftrightarrow \delta_{\psi(q)}$$

with q boxed in ψ.
Note: $\mathcal{S}^F$ is inconsistent since

$$\neg\Box\delta_{\neg\Box q} \leftrightarrow \delta_{\neg\Box q}$$

is an axiom of $\mathcal{S}^F$ and we can derive a contradiction using (T) and (Nec).

DML with two modal operators—Language

$\mathcal{L}^F_{\Box\blacksquare}$

The language $\mathcal{L}^F_{\Box\blacksquare}$ is the extension of the language $\mathcal{L}_{\Box\blacksquare}$ by fixed-points with respect to both modal operators.

$\mathcal{L}^F_{\Box} \cup \mathcal{L}^F_{\blacksquare}$

The language $\mathcal{L}^F_{\Box} \cup \mathcal{L}^F_{\blacksquare}$ is the closure of the formulas of $\mathcal{L}^F_{\Box}$ and $\mathcal{L}^F_{\blacksquare}$ under the formation rules for the boolean and modal operators.

Example

The fixed-point $\delta_{\neg\Box\blacksquare q}$ is a formula of $\mathcal{L}^F_{\Box\blacksquare}$ but not of $\mathcal{L}^F_{\Box} \cup \mathcal{L}^F_{\blacksquare}$ since the latter does not contain fixed-points for all formulas of $\mathcal{L}_{\Box\blacksquare}$ as it does not contain fixed-points for formulas containing both modal operators.

Explicating Reducibility

Consider an inconsistent $\mathcal{L}^F_{\square\blacksquare}$-theory T, such that the restriction of T to $\mathcal{L}_{\square\blacksquare}$ is consistent.

Explication

The inconsistency of T is reducible iff the restriction of T to $\mathcal{L}^F_{\square} \cup \mathcal{L}^F_{\blacksquare}$ is inconsistent.

Basically, that is to say that the theory is reducible, if we can already derive the inconsistency with a fixed-point for a formula containing only one modal operator.

Example: Fitch + Monotonicity is reducible

$(\blacksquare T)\blacksquare\varphi \to \varphi$; $(\Diamond\blacksquare In)\varphi \to \Diamond\blacksquare\varphi$; $(\Box Mon)\ \dfrac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$; $(\Box Nec)\ \dfrac{\varphi}{\Box\varphi}$.

Lemma

$(\blacksquare T), (\Box Mon), (\Diamond\blacksquare In) \vdash \Box\varphi \to \varphi$.

Proof.

1. $\blacksquare\varphi \to \varphi$ $\quad (\blacksquare T)$
2. $\neg\Box\neg\blacksquare\varphi \to \neg\Box\neg\varphi$ $\quad (\Box Mon)$
3. $\varphi \to \neg\Box\neg\blacksquare\varphi$ $\quad (\Diamond\blacksquare In)$
4. $\varphi \to \neg\Box\neg\varphi$ $\quad 2, 3$
5. $\Box\varphi \to \varphi$ $\quad 4$

$\Box$

Then we get the inconsistency with $Nec_\Box$ analogous to Montague's theorem using only the fixed-point $\delta_{\neg\Box q} \leftrightarrow \neg\Box\delta_{\neg\Box q}$.

No Future is not reducible

- We need to show that the diagonal extension of the modal logic of "No Future" is consistent in $\mathcal{L}^F_\Box \cup \mathcal{L}^F_\blacksquare$.

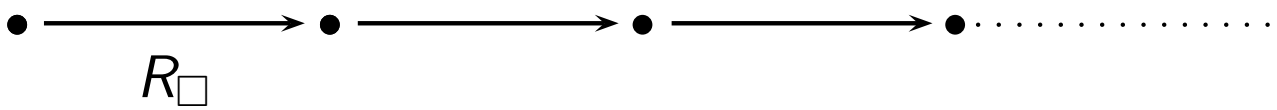
For this we will construct a model in three steps:

- First, we are giving a model for the theory KD^F in $\mathcal{L}_\Box$.
- Second, we define a new accessibility relation $R_\blacksquare$ satisfying the axioms of $\blacksquare$ as well as the interaction axioms
- Third, we expand the valuation to the $\mathcal{L}_\blacksquare$ fixed points.

Models of KD based on (ω, S)

Let KD be the normal modal logic containing $(D)\ \Box\neg\phi \to \neg\Box\phi$ and KD^F its fixed point extension.

- KD is valid on (ω, S).



Models of KD^F based on (ω, S)

Lemma

Every KD model $\mathfrak{M}$ based on (ω, S) can be expanded to a KD^F model $\mathfrak{M}'$ based on (ω, S).

(Sketch of a proof)
Let $\mathfrak{M} = (\omega, S, V)$ be a model of KD.

- We will expand V to a valuation V' of all fixed points $\delta_{\psi(q)}$.
- On the frame (ω, S) and for a fixed valuation V we can think of a formula $\psi(q)$ as inducing a function $\psi_q^V : 2^\omega \to 2^\omega$.
- To interpret $\delta_{\psi(q)}$ in such a way that $\delta_{\psi(q)} \leftrightarrow \psi(\ulcorner \delta_{\psi(q)} \urcorner)$ will hold, we need a countably infinite binary sequence s such that $\psi_q^V(s) = s$.

- Since q is guarded in ψ and the modal depth m of ψ is finite we have that for any sequence s the nth sequent of $\psi_q^V(s)$ is determined by the $(s)_{n+i}$ for $1 \leq i \leq m$ and independent of $(s)_n$.
- So if we consider the sequence s' which is identical to s for all sequents $(s)_{n+i}$ for $1 \leq i \leq m$ and with $(s')_n = (\psi_q^V(s))_n$. Generalizing this argument we get:

Lemma

For all V and for all $\psi(q)$ and for all $n \in \omega$ there is a sequence $s \in 2^\omega$ such that $(s)_i = (\psi_q^V(s))_i$ for all $i \leq n$.

- Now for any sequence s with the fixed point property for the first n entries we also have the fixed point property for all initial segments of s less then n.
- Then we can define a binary branching tree in which the paths correspond to initial segments that have the fixed point property for ψ_q^V.
- Since we have initial segments of arbitrary length the tree is infinite and by weak König's lemma we have an infinite path which gives us our desired infinite binary sequence which is a fixed point of ψ_q^V.
- This can be generalized such that we get our valuation V' and with it our model for KD^F based on (ω, S).

A model of interaction

$(\Box\blacklozenge In)\ \phi \rightarrow \Box\blacklozenge\phi$,
is a very simple Sahlquist formula and defines the first-order frame property:

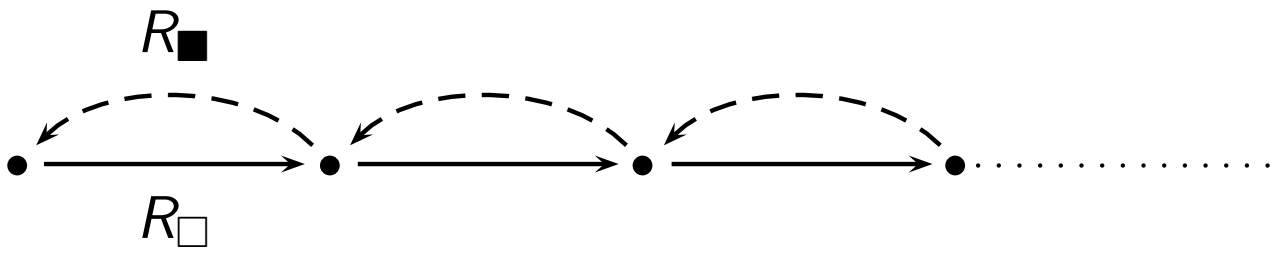
$$\forall w, v \in W(wR_{\Box}v \Rightarrow vR_{\blacksquare}w)$$

- Define $R_{\blacksquare}$ as $R_{\Box}^{-1}$, thus

$$wR_{\Box}v \Leftrightarrow vR_{\blacksquare}w$$

A picture

We obtain a frame of the following form:



Note that $R_{\blacksquare}$ is converse well-founded, i.e. for all $X \subseteq \omega$ there is a $w \in X$, such that for all $v \in X, \neg w R_{\blacksquare} v$!

The model—continued

- So far we have constructed a model for the language $\mathcal{L}^F_\Box \cup \mathcal{L}_\blacksquare$ which is a model of
  - $KD^F_\Box$
  - $(\blacksquare K)$ and $(\blacksquare Nec)$
  - $\phi \to \Box\blacklozenge\phi$, as for the underlying frame we have:

$$\forall w, v \in W(wR_\Box v \Rightarrow vR_\blacksquare w)$$

  - $\phi(\delta_{\phi(q)}) \leftrightarrow \delta_{\phi(q)}$ for all $\delta_{\phi(q)} \in \mathcal{L}^F_\Box$

The model—last step

- We need to extend the model to a model of $\mathcal{L}^F_\Box \cup \mathcal{L}^F_\blacksquare$ and provide a suitable interpretation for all fixed-point constants of $\mathcal{L}^F_\blacksquare$.
- But we only need to provide interpretations of fixed-points of formulas in which the $\Box$ operator does not occur.
- Any model of a modal language $\mathcal{L}_\Box$ based on a converse well-founded frame can be extended to a model of the language $\mathcal{L}^F_\Box$ in which all fixed-point axioms are true (cf. Smorynski 1985).
- We can extend the valuation function to provide a suitable interpretation of the fixed-point constants of $\mathcal{L}^F_\blacksquare$ along the lines of Smoryński (1985).

Sketch of Smorińsky's construction I

- We extend a valuation V to a valuation V^F in a stepwise process, such that $\delta_{\phi(q)} \leftrightarrow \phi(\delta_{\phi(q)})$ will be true at any world.
- We first define an interpretation of the fixed point constants $\delta_{\phi(q)}$ at the world 0.
- Since q is guarded in ϕ we can rewrite $\phi(\delta_{\phi(q)})$ as a truthfunctional decomposition $\psi(\blacksquare\chi_1(\delta_{\phi(q)}), ..., \blacksquare\chi_n(\delta_{\phi(q)}), \theta_1, ..., \theta_m)$, where $\delta_{\phi(q)}$ occurs only in the χ_i.

Sketch of Smorińsky's construction I

- At the world 0 no worlds are $R_{\blacksquare}$ accessible, so all the $\blacksquare\chi_i(\delta_{\phi(q)})$ will be true no matter what the value of $\delta_{\phi(q)}$ is.
- So we can stipulate $V^F(\delta_{\phi(q)}, 0) := 1$ iff $\langle \omega, suc \rangle, 0 \models \phi(\delta_{\phi(q)})[V]$.
- Since $V^F(\delta_{\phi(q)}, j)$ is only dependent on $V^F(\delta_{\phi(q)}, i)$ for $i < j$ we can define the valuation V^F inductively.

No Future is not reducible

- This shows that there is a model of No future in $\mathcal{L}^F_{\Box} \cup \mathcal{L}^F_{\blacksquare}$.
- No Future is not reducible.
- The fixed-point formula used in the derivation of the inconsistency necessarily contains both modalities.

Conclusion

There are genuinely new paradoxes arising in the setting of interacting modalities!

Extended Montague revisited

The $\mathcal{L}_{\Box\blacksquare}$ axioms used in No Future imply the $\mathcal{L}_{\Box\blacksquare}$ axioms used in the Extended Montague.

Lemma

Let $\mathcal{S}$ be a multimodal logic in $\mathcal{L}_{\Box\blacksquare}$.
If $(\Box Reg), (\blacksquare Reg), (\Box D), (\Box\blacklozenge In) \subseteq \mathcal{S}$, then $\mathcal{S} \vdash (\Box\blacksquare T)$.

$\Rightarrow$ Extended Montague is also not reducible.

- The proof and the principles used in Extended Montague seem to be very similar to the original Montague.
- Extended Montague appears to be 'structurally' reducible to the original Montague.
- There may be other relevant notions of reducibility!

Conclusion

- The syntactical treatment of modalities is especially suited for the interaction of modalities.
- The interaction of modalities may lead to inconsistency.
- DML is a way to investigate consistent subsystems.
- DML allows for an interesting notion of reducilibities of paradoxes.
- According to this notion there are irreducible paradoxes of interaction.

Thank you!