Faithful to classical logic

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Truth Be Told Again
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What counts as departure from classical logic?

A truth theory is nonclassical iff it's consequence relation isn't classical (for some sentences).

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Non classical logic cannot be easily contained.

In what follows I consider two ways of thinking about *grounded* truth.

One way is classical, the other is paraconsistent or paracomplete (or both).

They obviously differ on the liar sentence and other problematic sentences; but I'm more interested in other consequences and, in particular, in the differences concerning the unproblematic grounded sentences.

At least the proponents of nonclassical approaches claim that they don't mess with mathematics and classical logic on T-free sentences.

A set S is an SK-fixed-point iff it satisfies the following conditions:

- $(s = t) \in S$ iff s and t coincide in their values.
- $(s \neq t) \in S$ iff s and t differ in their values.
- $(\varphi \wedge \psi) \in S$ iff $\varphi, \psi \in S$.
- $(\neg(\varphi \wedge \psi)) \in S$ iff either $\neg\varphi \in S$ or $\neg\varphi \in S$.

  $\vdots$
- $T\ulcorner\varphi\urcorner \in S$ iff $\varphi \in S$.
- $\neg\, T\ulcorner\varphi\urcorner \in S$ iff $(\neg\varphi) \in S$.

NB. We do not have $(\neg\varphi) \in S$ iff $\varphi \notin S$.

Neither the liar sentence nor its negation can be in an SK-fixed-point.

The Kripke–Feferman theory KF is formulated in classical logic and describes SK-fixed-points as the extension of the truth predicate.
$(\mathbb{N}, S) \vDash$ KF iff S is an SK-fixed-point.

Partial Kripke–Feferman PKF is formulated in Strong Kleene logic (partial logic) and it's theorems are in all SK-fixed-points.

For SK-logic we need a system for SK-logic. Leon prefers Natural Deduction. For T-free formulae classical logic is retained.

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More formally: If all undischarged premisses is in an SK-fixed-point S, then the conclusion is in S; and if the negation of the conclusion is in S, then the negation of some premiss is in S.

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More formally: If all undischarged premisses is in an SK-fixed-point S, then the conclusion is in S; and if the negation of the conclusion is in S, then the negation of some premiss is in S.

$\nvdash \varphi \to \varphi$, but $\varphi \vdash \varphi$.

I concentrate on partial (SK) logic, nowadays often cold 'paracomplete'.

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If we define $\neg\varphi$ as $\varphi \to \bot$, we can keep almost all rules of ND for PKF such as:

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \qquad \frac{\bot}{\varphi} \qquad \frac{\varphi \qquad \varphi \to \psi}{\psi}$$

But we have to ditch conditionalization:

$$\frac{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}}{\varphi \to \psi} \qquad \text{but we can have} \qquad \frac{\varphi \vee \neg\varphi \qquad \begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}}{\varphi \to \psi}$$

In PKF We have all axioms of arithmetic in the usual form.

In PKF the induction rule is extended to the language with T:

$$\frac{\varphi(0) \qquad \begin{array}{c}\varphi(x)\\ \vdots \\ \varphi(Sx)\end{array}}{\forall x\, \varphi(x)}$$

In KF we have

$$\forall x, y(T(x \doteq y) \leftrightarrow \mathrm{val}(x) = \mathrm{val}(y))$$

In PKF we have a rule and its inverse:

$$\frac{\mathrm{val}(x) = \mathrm{val}(y)}{T\, x \doteq y}$$

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$$\forall x, y\big(\mathrm{ClTerm}(x) \wedge \mathrm{ClTerm}(y) \rightarrow (T(\dot{\neg} x \doteq y) \leftrightarrow \mathrm{val}(x) \neq \mathrm{val}(y))\big)$$

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In KF we have

$$\forall x \forall y \big((T\,x \wedge T\,y) \leftrightarrow T(x \underset{\cdot}{\wedge} y) \big)$$

In PKF we have the following rule and its inverse:

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In KF we have

$$\forall x\big(\,\mathrm{ClTerm}(x) \to (T\,\underset{\cdot}{T}x \leftrightarrow T\,\mathrm{val}(x))\big)$$

In PKF we have the following rule and its inverse:

$$\frac{\mathrm{ClTerm}(x) \qquad T\,\mathrm{val}(x)}{T\,\underset{\cdot}{T}x}$$

As long as we can prove $\varphi \vee \neg\varphi$ the formulae φ behaves classically. We can prove it for arithmetical sentences and iterations of truth.

In PKF we should get the *axioms* at least for the 'good', grounded truths from the corresponding rules. We do have for instance:

$$\mathrm{PKF} \vdash \forall x, y\big(\mathrm{ClTerm}(x) \wedge \mathrm{ClTerm}(y) \rightarrow (T \dot{\neg} x \dot{=} y \leftrightarrow \mathrm{val}(x) \neq \mathrm{val}(y))\big)$$

And we also have:

$$\mathrm{PKF} \vdash \forall x \forall y\big(\mathrm{Sent}_A(x) \wedge \mathrm{Sent}_A(y) \rightarrow (T(x \dot{\wedge} y) \leftrightarrow T\,x \wedge T\,y)\big)$$

where $\mathrm{Sent}_A(x)$ says that x is a T-free sentence.

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Now let's climb up through all grounded layers on truth.

Hence, as expected, in a sense truth doesn't seem to behave differently on the classical and the nonclassical account.

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Perhaps, as long as we don't consider pathological sentences, classical and nonclassical truth coincide; and the classical and nonclassical logician differ only in the way they think about liar sentences and other pathological sentences: The nonclassical logician would like to keep full transparency, the classical logician the full rule of conditionalization.

But now comes the surprise: The disagreement between KF and PKF is not on truth, but rather on how much transfinite induction one can get.

In KF we can show truth iterations up to (but excluding)

$$\epsilon_0 = \omega^{\omega^{\vdots^{\omega}}}$$

many levels are grounded. In other words, KF proves that sentences with up to ϵ_0 many truth iterations are truth-determinate:

$$KF \vdash \forall x \left(\mathrm{Sent}_{<\epsilon_0}(x) \rightarrow Tx \vee \neg Tx\right)$$

In PKF we get only the levels below ω^{ω}.

KF and PKF don't differ so much in their assumptions on truth; they agree on the principles for grounded truths. But since they don't agree about which truths are grounded, they also prove different theorems about grounded truths; and, consequently, they differ even on T-free sentences.

KF and PKF differ on the *mathematical schemata* one can prove and *that* makes PKF weaker than KF, because KF can see of more sentences that they are grounded.

By going from classical logic to paracomplete or paraconsistent logic you don't mess up your reasoning about (grounded) truth, but about mathematics.

The Kripke–Feferman theory

1. $\forall x, y\big(\mathrm{ClTerm}(x) \wedge \mathrm{ClTerm}(y) \rightarrow (Tx \doteq y \leftrightarrow \mathrm{val}(x) = \mathrm{val}(y))\big)$

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3. $\forall x\big(\mathrm{Sent}(x) \rightarrow (T\, \dot{\neg}\dot{\neg} x \leftrightarrow T\, x)\big)$

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3. $\forall x\big(\mathrm{Sent}(x) \rightarrow (T\, \dot{\neg}\dot{\neg} x \leftrightarrow T\, x)\big)$
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7. $\forall x \forall y\big(\mathrm{Sent}(x) \wedge \mathrm{Sent}(y) \rightarrow (T \dot{\neg}(x \vee y) \leftrightarrow T \dot{\neg} x \wedge T \dot{\neg} y)\big)$

The Kripke–Feferman theory

8. $\forall v \forall x \big(\mathrm{Var}(v) \wedge \mathrm{For}(x, v) \rightarrow (T \underset{\cdot}{\forall} v x \leftrightarrow \forall y\, T\, x(\dot{y}/v))\big)$

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Cons $\forall x \big(\mathrm{Sent}(x) \rightarrow \neg(T\, x \wedge T \dot{\neg} x)\big)$

PKF – Strong Kleene logic

In the following the provide a direct formalization of Kripke's theory in partial (Strong Kleene) logic formulated in a sequent calculus. We call it PKF.

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Sequents are conceived as given by a pair Γ and Δ of finite sets of formulas. The sequent is written as $\Gamma \Rightarrow \Delta$. If Γ and Δ are sets of sentences and $\Gamma \Rightarrow \Delta$ is derivable, our system PKF is sound in the sense that if all sentences in Γ are true in a partial model, then at least one sentence in Δ is true in that model, and if all sentences in Δ are false in a partial model, then at least one sentence in Γ is false in the model.

PKF – Structural rules and initial sequents

All sequents of one of the following forms are initial sequents:

$$\Gamma \Rightarrow \Delta, \text{ where } \Gamma \cap \Delta \neq \emptyset \qquad \text{(IN)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta} \qquad \text{(weakening 1)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta} \qquad \text{(weakening 2)}$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{(cut)}$$

PKF – Laws of truth values

$\top$ is the sentence 0=0, $\bot$ the sentence 0 = 1 and λ is the liar sentence (that is a sentence that is 'gappy' under the intended interpretation). The following sequents are then initial sequents:

$$\Rightarrow \top \qquad (\top\text{-sequent})$$

$$\bot \Rightarrow \qquad (\bot\text{-sequent})$$

$$\lambda \Leftrightarrow \neg\lambda \qquad (\lambda\text{-sequents})$$

In the last line and in the following the double arrow indicates that both, $\lambda \Rightarrow \neg\lambda$ and $\neg\lambda \Rightarrow \lambda$ are initial sequents. This convention will also be applied below.

PKF – Laws of negation

If Γ is a set of sentences, $\neg\Gamma$ designates the set of all negations of sentences in Γ.

$$\frac{\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \neg\Gamma} \qquad (\neg\text{-rule})$$

$$\varphi \Leftrightarrow \neg\neg\varphi \qquad (\neg\neg\text{-sequents})$$

PKF – Laws of $\vee$ and $\wedge$

$$\varphi, \psi \Rightarrow \varphi \wedge \psi \tag{$\wedge 1$}$$

$$\varphi \wedge \psi \Rightarrow \varphi \tag{$\wedge 2$}$$

$$\varphi \wedge \psi \Rightarrow \psi \tag{$\wedge 3$}$$

$$\varphi \vee \psi \Rightarrow \varphi, \psi \tag{$\vee 1$}$$

$$\varphi \Rightarrow \varphi \vee \psi \tag{$\vee 2$}$$

$$\psi \Rightarrow \varphi \vee \psi \tag{$\vee 3$}$$

PKF – Laws of quantifiers

$$\forall x \varphi \Rightarrow \varphi(t/x) \qquad (\forall 1)$$

$$\varphi(t/x) \Rightarrow \exists x \varphi \qquad (\exists 1)$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \forall x \varphi, \Delta} \quad x \text{ not free in lower sequent} \qquad (\forall 2)$$

$$\frac{\Delta, \varphi \Rightarrow \Gamma}{\Delta, \exists x \varphi \Rightarrow \Gamma} \quad x \text{ not free in lower sequent} \qquad (\exists 2)$$

PKF – Laws of Identity

We also add initial sequents for identity for arbitrary terms s and t:

$$\Rightarrow t = t \tag{=1}$$

$$s = t, \varphi(s/x) \Rightarrow \varphi(t/x) \tag{=2}$$

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Moreover we have the following induction rule:

$$\frac{\Gamma, \varphi(x) \Rightarrow \varphi(x+1), \Delta}{\Gamma, \varphi(\overline{0}), \Rightarrow \varphi(t), \Delta} \qquad \text{(IND)}$$

PKF – Truth

PKF1 (i) $\mathrm{ClTerm}(x), \mathrm{ClTerm}(y), \mathrm{val}(x) = \mathrm{val}(y) \Rightarrow T\, x \doteq y$

(ii) $\mathrm{ClTerm}(x), \mathrm{ClTerm}(y), T\, x \doteq y \Rightarrow \mathrm{val}(x) = \mathrm{val}(y)$

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PKF2 (i) $\mathrm{Sent}(x), \mathrm{Sent}(y), T\, x \wedge T\, y \Rightarrow T(x \underset{\cdot}{\wedge} y)$

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PKF3 (i) $\mathrm{Sent}(x), \mathrm{Sent}(y), T\, x \vee T\, y \Rightarrow T(x \underset{\cdot}{\vee} y)$

(ii) $\mathrm{Sent}(x), \mathrm{Sent}(y), T(x \underset{\cdot}{\vee} y) \Rightarrow T\, x \vee T\, y$

PKF – Truth

PKF1 (i) $\mathrm{ClTerm}(x), \mathrm{ClTerm}(y), \mathrm{val}(x) = \mathrm{val}(y) \Rightarrow T\, x \underset{\cdot}{=} y$

(ii) $\mathrm{ClTerm}(x), \mathrm{ClTerm}(y), T\, x \underset{\cdot}{=} y \Rightarrow \mathrm{val}(x) = \mathrm{val}(y)$

PKF2 (i) $\mathrm{Sent}(x), \mathrm{Sent}(y), T\, x \wedge T\, y \Rightarrow T(x \underset{\cdot}{\wedge} y)$

(ii) $\mathrm{Sent}(x), \mathrm{Sent}(y), T(x \underset{\cdot}{\wedge} y) \Rightarrow T\, x \wedge T\, y$

PKF3 (i) $\mathrm{Sent}(x), \mathrm{Sent}(y), T\, x \vee T\, y \Rightarrow T(x \underset{\cdot}{\vee} y)$

(ii) $\mathrm{Sent}(x), \mathrm{Sent}(y), T(x \underset{\cdot}{\vee} y) \Rightarrow T\, x \vee T\, y$

PKF4 (i) $\mathrm{Var}(v), \mathrm{For}(x, v), \forall y\, T\, x(\dot{y}/v)) \Rightarrow T \underset{\cdot}{\forall} v x$

(ii) $\mathrm{Var}(v), \mathrm{For}(x, v), T \underset{\cdot}{\forall} v x \Rightarrow \forall y\, T\, x(\dot{y}/v))$

PKF – Truth

PKF5 (i) $\mathrm{Var}(v), \mathrm{For}(x, v), \exists y\, T\, x(\dot{y}/v)) \Rightarrow T\, \dot{\exists} v x$

(ii) $\mathrm{Var}(v), \mathrm{For}(x, v), T\, \dot{\exists} v x \Rightarrow \exists y\, T\, x(\dot{y}/v))$

PKF – Truth

PKF5 (i) $\mathrm{Var}(v), \mathrm{For}(x, v), \exists y\, T\, x(\dot{y}/v)) \Rightarrow T\, \dot{\exists} v x$

(ii) $\mathrm{Var}(v), \mathrm{For}(x, v), T\, \dot{\exists} v x \Rightarrow \exists y\, T\, x(\dot{y}/v))$

PKF6 (i) $\mathrm{ClTerm}(x), T\, \mathrm{val}(x) \Rightarrow T\, \dot{T} x$

(ii) $\mathrm{ClTerm}(x), T\, \dot{T} x \Rightarrow T\, \mathrm{val}(x)$

PKF – Truth

PKF5 (i) $\mathrm{Var}(v), \mathrm{For}(x, v), \exists y\, T\, x(\dot{y}/v)) \Rightarrow T\, \dot{\exists} v x$

(ii) $\mathrm{Var}(v), \mathrm{For}(x, v), T\, \dot{\exists} v x \Rightarrow \exists y\, T\, x(\dot{y}/v))$

PKF6 (i) $\mathrm{ClTerm}(x), T\, \mathrm{val}(x) \Rightarrow T\, \dot{T} x$

(ii) $\mathrm{ClTerm}(x), T\, \dot{T} x \Rightarrow T\, \mathrm{val}(x)$

PKF7 (i) $\mathrm{Sent}(x), \neg\, T\, x \Rightarrow T\, \dot{\neg} x$

(ii) $\mathrm{Sent}(x), T\, \dot{\neg} x \Rightarrow \neg\, T\, x$

PKF – Truth

PKF5 (i) $\mathrm{Var}(v), \mathrm{For}(x, v), \exists y\, T\, x(\dot{y}/v)) \Rightarrow T\, \dot{\exists} v x$

(ii) $\mathrm{Var}(v), \mathrm{For}(x, v), T\, \dot{\exists} v x \Rightarrow \exists y\, T\, x(\dot{y}/v))$

PKF6 (i) $\mathrm{ClTerm}(x), T\, \mathrm{val}(x) \Rightarrow T\, \dot{T} x$

(ii) $\mathrm{ClTerm}(x), T\, \dot{T} x \Rightarrow T\, \mathrm{val}(x)$

PKF7 (i) $\mathrm{Sent}(x), \neg\, T\, x \Rightarrow T\, \dot{\neg} x$

(ii) $\mathrm{Sent}(x), T\, \dot{\neg} x \Rightarrow \neg\, T\, x$

PKF8 $T\, x \Rightarrow \mathrm{Sent}(x)$