Deflating Truth

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Truth be Told again
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Conservativity and truth

Proof-theoretic techniques

Expanding the realm of ordinal analysis
Outline

Conservativity and truth

Proof-theoretic techniques

Expanding the realm of ordinal analysis
\( \mathcal{L}_T \) extends the language \( \mathcal{L} \) of arithmetic by a fresh unary predicate T.

All our theories of truth have PA as background theory.

Depending on their axiomatisation, theories of truth over PA may have different proof-theoretic strength:

- \( T(PA) \) and \( CT^- \) are conservative extensions of PA.
- CT is a conservative extension of \( PA + TI(<\epsilon\epsilon_0) \).
- FS is a conservative extension of \( PA + TI(<\varphi20) \).
- KF is a conservative extension of \( PA + TI(<\varphi\epsilon_0\alpha) \).
- VF is a conservative extension of \( PA + TI(<\vartheta\epsilon\Omega+1) \).
Sequent calculi for truth

Use a Gentzen-style sequent calculus: $\Gamma \Rightarrow^\alpha \Delta$:

1. $\Gamma, \Delta$ finite sets of formulæ,
2. $\alpha$ ordinal bound.

Only consider “strong” theories, so use $\omega$-logic (e.g. PA$\omega$).

Rules of inference for background theory. E.g.

\[
\begin{align*}
(\wedge R) & \quad \frac{\Gamma \Rightarrow \Delta, \phi_0 \quad \Gamma \Rightarrow \Delta, \phi_1}{\Gamma \Rightarrow \Delta, \phi_0 \wedge \phi_1} \\
(\wedge L) & \quad \frac{\Gamma, \phi_i \Rightarrow \Delta}{\Gamma, \phi_0 \wedge \phi_1 \Rightarrow \Delta} \\
(\forall R) & \quad \frac{\Gamma \Rightarrow \Delta, \phi(\bar{n}) \text{ for every } n < \omega}{\Gamma \Rightarrow \Delta, \forall x \phi} \\
(\forall L) & \quad \frac{\Gamma, \phi(\bar{n}) \Rightarrow \Delta}{\Gamma, \forall x \phi \Rightarrow \Delta} \\
(\neg R) & \quad \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \phi} \\
(\neg L) & \quad \frac{\Gamma \Rightarrow \Delta, \phi}{\Gamma, \neg \phi \Rightarrow \Delta}
\end{align*}
\]

also $(\lor L), (\lor R), (\exists L), (\exists R), (\rightarrow L), (\rightarrow R)$
Rules for truth

Formulate the truth axioms as rules of inference.

\[
\begin{align*}
\text{(\&}_T R) & \quad \frac{\Gamma \Rightarrow \Delta, T s_0 \quad \Gamma \Rightarrow \Delta, T s_1}{\Gamma \Rightarrow \Delta, T (s_0 \& s_1)} \\
\text{(\&}_T L) & \quad \frac{\Gamma, T s_i \Rightarrow \Delta}{\Gamma, T (s_0 \& s_1) \Rightarrow \Delta} \\
\text{(\forall}_T R) & \quad \frac{\Gamma \Rightarrow \Delta, T \nabla \phi(\bar{n}) \nabla \text{every } n < \omega}{\Gamma \Rightarrow \Delta, T \nabla \forall x \phi \nabla} \\
\text{(\forall}_T L) & \quad \frac{\Gamma, T \nabla \phi(s) \nabla \Rightarrow \Delta}{\Gamma, T \nabla \forall x \phi \nabla \Rightarrow \Delta} \\
\text{(-}_T R) & \quad \frac{\Gamma, T s \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, T \neg s} \\
\text{(-}_T L) & \quad \frac{\Gamma \Rightarrow \Delta, T s}{\Gamma, T \neg s \Rightarrow \Delta} \\
\text{(T}_R) & \quad \frac{\Gamma \Rightarrow \Delta, T s}{\Gamma \Rightarrow \Delta, T (T s)} \\
\text{(T}_L) & \quad \frac{\Gamma, T s \Rightarrow \Delta}{\Gamma, T (T s) \Rightarrow \Delta}
\end{align*}
\]

and similarly for $\exists$, $\vee$, $\rightarrow$ and $=$.

For a given set Th of truth rules, we want to show:

If $\Gamma \Rightarrow^\alpha \Delta$ is derivable in Th and $\Gamma \cup \Delta$ is truth-free,
then $\Gamma \Rightarrow^\beta \Delta$ is derivable in PA$_\omega$ for some determined $\beta$. 
Partial cut elimination

Theorem
Let $Th$ be some set of sequent rules above. Then if $\Gamma \Rightarrow^\alpha \Delta$ is derivable in $Th$, then $\Gamma \Rightarrow \Delta$ has a derivation in which all cuts are on the atomic truth predicate only; its height is bounded by $2^\alpha_n$ for some $n$ where $2^\alpha_{n+1} = 2^{2^\alpha_n}$ and $2^\alpha_0 = \alpha$.

Question
Given $\Gamma \Rightarrow^\alpha \Delta$ is there a cut-free derivation of $\Gamma \Rightarrow^\beta \Delta$ for some $\beta$? If so, for which $\beta$?
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Cut elimination
Compositional truth

- $\text{CT}_\omega$ comprises all rules of $\text{PA}_\omega$ plus the compositional rules for truth ($(\land T)R$, $(\land TL)$, $(\forall T)R$, $(\forall TL)$ etc.)
- No rules of explicit self-reference.
- $\text{CT}_\omega$ permits full cut elimination. Define two rank functions $|.|_0$ and $|.|_1$ on $L_T$ as follows:
  - $|R(t_1, \ldots, t_n)|_i = 0$, $|T|_i = \omega \cdot i$
  - $|\neg \phi|_i = |Qv\phi|_i = |\phi|_i + 1$, $|\phi \diamond \psi|_i = \max\{|\phi|_i, |\psi|_i\} + 1$
- P.c.e. entails every derivation can be turned into one in which cut formulæ have $|.|_1$-rank $\omega$.
- A cut on $T$ can then be eliminated by induction on $|s^N|_0$. 
Language: $\mathcal{L}_\delta = \mathcal{L} \cup \{T_\xi \mid \xi < \delta\}$.

Axioms of RT$^{<\kappa}$ include the axioms of CT for each predicate $T_\lambda$ with $\lambda < \kappa$, plus:

\[
\begin{align*}
&T_{\delta R} \quad \frac{\Gamma \Rightarrow \Delta, T_\gamma s}{\Gamma \Rightarrow \Delta, T_\delta \upharpoonright T_\gamma s} \quad \gamma < \delta \\
&T_{\delta L} \quad \frac{\Gamma, T_\gamma s \Rightarrow \Delta}{\Gamma, T_\delta \upharpoonright T_\gamma s \Rightarrow \Delta} \quad \gamma < \delta
\end{align*}
\]
Cut elimination
Ramified truth

- $\text{RT}_{\omega}^{<\kappa}$ also permits cut elimination.
- Replace all the truth rules by axiom $\Gamma, T_\lambda\sigma \Rightarrow \Delta, T_\lambda\sigma$ and

\[
\frac{\Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, T_\lambda\phi} \quad \text{lvl}(\phi) < \lambda
\]

\[
\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, T_\lambda\phi \Rightarrow \Delta} \quad \text{lvl}(\phi) < \lambda
\]

for $\lambda < \kappa$.

- Define rank functions $|.|$ by:

\[
|R(t_1, \ldots, t_n)| = 0, \quad |T_\gamma\sigma| = \omega \cdot \gamma,
\]

\[
|\neg\phi| = |Qv\phi| = |\phi| + 1, \quad |\phi \diamond \psi| = \max\{|\phi|, |\psi|\} + 1,
\]

- If $\text{lvl}(\phi) < \lambda < \kappa$ then $|\phi| < \omega \cdot \delta + \omega \leq |T_\lambda\phi|$.
Formalised model constructions
Friedman-Sheard theory of truth

- FS extends CT by finitely iterated truth:

\[
\begin{align*}
\text{Nec} & \quad \frac{\phi}{T^n \phi} \\
\text{co-Nec} & \quad \frac{T^n \phi}{\phi}
\end{align*}
\]

- To form FS\(_\omega\), add these rule to CT\(_\omega\):

\[
\begin{align*}
\text{(Nec}_m) & \quad \frac{\emptyset \Rightarrow_m \phi}{\Gamma \Rightarrow_{m+1} \Delta, Ts} \quad \text{if } s^N = \neg \phi \\
\text{(co-Nec}_m) & \quad \frac{\emptyset \Rightarrow_m T^n \phi}{\Gamma \Rightarrow_{m} \Delta, \phi} \quad \text{if } s^N = \neg \phi
\end{align*}
\]

- Adding co-Nec ruins partial cut elimination.
Formalised model constructions

**Theorem**
Suppose $\Gamma \Rightarrow_\alpha^m \Delta$ is derivable in $\text{FS}_\omega$ and $\Gamma \cup \Delta$ is truth-free. Then $\Gamma \Rightarrow \Delta$ has a derivation in $\text{PA}_\omega$ with height bounded by $\epsilon_\alpha^m$ for some $m$, where $\epsilon_\alpha^0 = \alpha$ and $\epsilon_\alpha^{m+1} = \epsilon_\epsilon^m$.

**Proof.**

- Define $\mathcal{M}_0^X = \langle \mathbb{N}, X \rangle$ and $\mathcal{M}_{n+1}^X = \langle \mathbb{N}, \{ \neg \phi \upharpoonright \mathcal{M}_n^X \models \phi \}\rangle$.
- Notice that (*) $\Gamma \Rightarrow_\alpha^m \Delta$ implies $\mathcal{M}_m^X \models \wedge \Gamma \rightarrow \vee \Delta$ for every $X \subseteq \omega$.
- The above proof can be formalised in systems in which the operation $X \mapsto \mathcal{M}_0^X$ is definable.
- $\{ \neg \phi \upharpoonright \mathcal{M}_0^X \models \phi \}$ is recursive in $X^{(\omega)}$, so (*) is provable in:
  - $\text{ACA} + \forall X \exists Y (Y = X^{(\omega)})$ in general; or
  - $\text{ACA}_0 + \forall X \exists Y (Y = X^{(\omega)})$ for “meta” $m$.
- $\text{ACA}_0 + \forall X \exists Y (Y = X^{(\omega)})$ conservatively extends $\text{PA} + \{ \text{TI}(\prec \epsilon_0^m) \mid m < \omega \}$. □
Asymmetric interpretations
Kripke-Feferman truth

\[ \text{KF}_\omega \text{ extends } \text{CT}_\omega - \{(\neg_T R), (\neg_T L)\} \text{ by strong self-applicability:} \]

\[
\begin{align*}
\text{(T}_1\text{R}) & \quad \Gamma \Rightarrow \Delta, Tr \\
& \quad \Gamma \Rightarrow \Delta, TTr \\
\text{(T}_2\text{R}) & \quad \Gamma \Rightarrow \Delta, T\neg r \\
& \quad \Gamma \Rightarrow \Delta, T\neg Tr \\
\text{(T}_1\text{L}) & \quad \Gamma, Tr \Rightarrow \Delta \\
& \quad \Gamma, TTr \Rightarrow \Delta \\
\text{(T}_2\text{L}) & \quad \Gamma, T\neg r \Rightarrow \Delta \\
& \quad \Gamma, T\neg Tr \Rightarrow \Delta
\end{align*}
\]

\[ \text{KF}_\omega \text{ permits partial cut elimination.} \]

\[ \text{There is no canonical measure of complexity that covers both (T}_1\text{R) and (T}_1\text{L).} \]
Asymmetric interpretations

Define, by transfinite induction, sets $S^+_\alpha$, $S^-_\alpha$:

- $S^+_0 = S^-_0 = \emptyset$,
- $S^+_{\alpha+1} = \{ \Gamma \phi^- \mid \langle \mathbb{N}, S^+_{\alpha}, S^{-}_{\alpha} \rangle \models_{SK} \phi \}$,
- $S^-_{\alpha+1} = \{ \Gamma \phi^- \mid \langle \mathbb{N}, S^+_{\alpha}, S^{-}_{\alpha} \rangle \models_{SK} \overline{\phi} \}$,
- $S^+_\lambda = \bigcup_{\alpha < \lambda} S^+_\alpha$, $S^-_\lambda = \bigcup_{\alpha < \lambda} S^-_\alpha$.

For $\phi$ in cnf, define $\phi^{(\alpha, \beta)}$:

- $(Ts)^{(\alpha, \beta)}$ iff $s^\mathbb{N} \in S^+_{\alpha}$
- $(\neg Ts)^{(\alpha, \beta)}$ iff $s^\mathbb{N} \not\in S^-_{\beta}$
- $(\phi \land \psi)^{(\alpha, \beta)}$ iff $\phi^{(\alpha, \beta)}$ and $\psi^{(\alpha, \beta)}$
- $(\forall x \phi)^{(\alpha, \beta)}$ iff for all $n$, $\phi^\bar{n}^{(\alpha, \beta)}$

Lemma (Cantini)

If $\Gamma \Rightarrow^\alpha \Delta$ is derivable in $\text{KF}_\omega$ then for every $\beta$, $(\bigvee \Gamma \lor \bigvee \Delta)^{\beta+\omega^\alpha, \beta}$ is true.
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Axiomatisable concepts

Truth-theoretic principles can axiomatise other meta-theoretic concepts.

- **Provability:**
  - $P\phi \land P(\phi \rightarrow \psi) \rightarrow P\psi$, 
  - from $\phi$ infer $P\phi$, 
  - $P\phi \rightarrow PP\phi$.

- **Knowledge:**
  - $K\phi \land K(\phi \rightarrow \psi) \rightarrow K\psi$, 
  - from $\phi$ infer $K\phi$, 
  - $K\phi \rightarrow \phi$.

- **Intuitionistic truth:**
  - Compositional axioms for $\rightarrow, \land, \bot, \forall$, etc.
  - $T(\phi \lor \psi) \leftrightarrow T\phi \lor T\psi$ and $T(\exists x \phi) \leftrightarrow \exists x T(\phi(x))$, 
  - Some self-reference?

- **Provability in $\omega$-logic with reflection:**
  - Axioms of provability, 
  - $\forall x P(\phi x) \rightarrow P(\forall x \phi)$, 
  - from $P\phi$ infer $\phi$, or $PP\phi \rightarrow P\phi$, or $P\phi \rightarrow \phi$. 
Consider the following intuitionistic theory $S$ extending $\text{HA} + \text{Ind}_{\mathcal{L}_P}$:

1. $\phi(x) \rightarrow T\phi(\dot{x})$ for $\phi$ in $\Delta_0$

2. $\forall\phi\forall\psi(P\phi \land P(\phi \rightarrow \psi) \rightarrow P\psi)$

3. Compositional axioms for $\forall$, $\exists$, $\lor$ and $\land$:
   \[
   \forall x P(\phi(x)) \leftrightarrow P(\forall x \phi) \quad P(\phi \lor \psi) \leftrightarrow P\phi \lor P\psi
   \]
   \[
   \exists x P(\phi(x)) \leftrightarrow P(\exists x \phi) \quad P(\phi \land \psi) \leftrightarrow P\phi \land P\psi
   \]

4. Internal reflection $\forall\phi(P\phi \rightarrow P\phi)$

5. ‘from $\phi$ infer $P\phi$’ and ‘from $P\phi$ infer $\phi$’
A theory of provability with weak reflection
Outline of a proof-theoretic analysis

Steps in the proof-theoretic analysis of S:
1. Present S as a sequent calculus
2. Informal model construction
3. Formalise the consistency proof:
   ▶ Asymmetric interpretation
   ▶ Interpreting derivability as arithmetical truth
4. Embed theorems of S into a theory of intuitionistic truth
5. Perform cut elimination
The informal model construction

S as a sequent calculus:

\begin{align*}
& (\wedge \text{P} \text{R}), (\wedge \text{P} \text{L}), (\vee \text{P} \text{R}), (\vee \text{P} \text{L}) & \text{(\forall \text{P} \text{R}), (\forall \text{P} \text{L}), (\exists \text{P} \text{R}), (\exists \text{P} \text{L})} \\
\end{align*}

\[
\frac{\Gamma \Rightarrow_m \text{P}\phi \quad \Gamma, \text{P}\psi \Rightarrow_m \chi}{\Gamma, \text{P}(\phi \rightarrow \psi) \Rightarrow_m \chi}
\]

\[
\frac{\Gamma, \text{Ps} \Rightarrow_m \phi}{\Gamma, \text{PPs} \Rightarrow_m \phi}
\]

\[
\frac{\varnothing \Rightarrow_m \phi}{\Gamma \Rightarrow_n \text{P}^\Gamma \phi^{-}} \quad m < n
\]

Then

\begin{itemize}
\item $\Rightarrow_{m+1}$ has the disjunction and existence property, and
\item If $\Gamma \Rightarrow_{m+1} \phi$ is derivable then
  \[
  \langle \mathbb{N}, \{\neg\psi^{-} | \varnothing \Rightarrow_m \psi} \rangle \models (\wedge \Gamma) \rightarrow \phi.
  \]
\item $\varnothing \Rightarrow_{m+1} \text{P}\phi$ implies $\varnothing \Rightarrow_m \phi$.
\end{itemize}

So if $S \vdash \phi$ then $\varnothing \Rightarrow_m \phi$ for some $m$.

In which system can the soundness proof be formalised?
Formalising the model construction

Asymmetric interpretations

We make use of asymmetric interpretations:

For \( m < \omega, \alpha \) and \( \beta \), define \( A^{(\alpha,\beta,m)} \) by:

\[
(P\phi)^{\alpha,\beta,m} \iff \exists \gamma < \varphi_m \alpha \exists n < m (\emptyset \Rightarrow^\gamma n \phi)
\]

\[
(A \rightarrow B)^{\alpha,\beta,m} \iff (A^{\beta,\alpha,m} \rightarrow B^{\alpha,\beta,m}) \land \exists \gamma < \varphi_m \alpha (A \Rightarrow^\gamma_m B)
\]

\[
(A \circ B)^{\alpha,\beta,m} \iff (A^{\alpha,\beta,m} \circ B^{\alpha,\beta,m}) \quad (\circ \in \{\land, \lor\})
\]

\[
(QxA)^{\alpha,\beta,m} \iff (Qn A(n)^{\alpha,\beta,m}) \quad Q \in \{\forall, \exists\}
\]

Lemma

If \( \Gamma \Rightarrow^\alpha_m \phi \) then for every \( \beta \), \( \mathbb{N} \models (\land \Gamma \rightarrow \phi)^{(\beta+\omega^\alpha,\beta,m)} \).

Question

How do we formalise \( \mathbb{N} \models \phi^{(\alpha,\beta,m)} \)?
Formalising the model construction
Formalising truth

**Question**

*How do we formalise* $\mathbb{N} \models \phi^{(\alpha, \beta, m)}$?

**Answer**

*Use a truth predicate:*

$\Gamma \vDash^\alpha_m \phi$ *implies* $\text{CT}^i + \text{TI}(\varphi_{m+1}) \vdash \forall \beta \ T(\Gamma (\land \Gamma \rightarrow \phi)^{(\omega^\alpha, \beta, m) \downarrow})$.

Thus, if $\Gamma \vDash^\alpha_m \phi$ is derivable in $S^\omega$ and $\Gamma, \phi$ are truth free,

1. $\emptyset \vDash^\beta \ T(\land \Gamma \rightarrow \phi)$ *is derivable in* $\text{CT}^i_{\omega}$ *for some* $\beta < \varphi_{\omega^0}$,
2. $\Gamma \vDash^\beta \phi$ *is derivable in* $\text{CT}^i_{\omega}$ *for some* $\beta < \varphi_{\omega^0}$,
3. $\Gamma \vDash \phi$ *is cut-free derivable in* $\text{CT}^i_{\omega}$ *with height* $\varphi_{\omega^0}$,
4. $\Gamma \vDash \phi$ *is derivable in* $\text{HA} + \text{TI}(\varphi_{\omega^0})$.

*So* $S$ *is a conservative extension of* $\text{HA} + \text{TI}(\varphi_{\omega^0})$.  \(\square\)