Deflating Truth

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Truth be Told again
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Outline

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Conservativity and truth

Proof-theoretic techniques

Expanding the realm of ordinal analysis

Conservative over what?

$\mathcal{L}_{\mathrm{T}}$ extends the language $\mathcal{L}$ of arithmetic by a fresh unary predicate T.

All our theories of truth have PA as background theory.

Depending on their axiomatisation, theories of truth over PA may have different proof-theoretic strength:

- T(PA) and CT^{-} are conservative extensions of PA.
- CT is a conservative extension of $\mathrm{PA} + \mathrm{TI}(<\epsilon_{\epsilon_0})$.
- FS is a conservative extension of $\mathrm{PA} + \mathrm{TI}(<\varphi 20)$.
- KF is a conservative extension of $\mathrm{PA} + \mathrm{TI}(<\varphi\epsilon_0 0)$.
- VF is a conservative extension of $\mathrm{PA} + \mathrm{TI}(<\vartheta\epsilon_{\Omega+1})$.

Sequent calculi for truth

Use a Gentzen-style sequent calculus: $\Gamma \Rightarrow^{\alpha} \Delta$:

1. Γ, Δ finite sets of formulæ,
2. α ordinal bound.

Only consider "strong" theories, so use ω-logic (e.g. PA_ω).

Rules of inference for background theory. E.g.

$$(\wedge\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, \phi_0 \qquad \Gamma \Rightarrow \Delta, \phi_1}{\Gamma \Rightarrow \Delta, \phi_0 \wedge \phi_1} \qquad (\wedge\mathrm{L})\ \frac{\Gamma, \phi_i \Rightarrow \Delta}{\Gamma, \phi_0 \wedge \phi_1 \Rightarrow \Delta}$$

$$(\forall\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, \phi(\bar{n}) \text{ for every } n < \omega}{\Gamma \Rightarrow \Delta, \forall x \phi} \qquad (\forall\mathrm{L})\ \frac{\Gamma, \phi(\bar{n}) \Rightarrow \Delta}{\Gamma, \forall x \phi \Rightarrow \Delta}$$

$$(\neg\mathrm{R})\ \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\phi} \qquad (\neg\mathrm{L})\ \frac{\Gamma \Rightarrow \Delta, \phi}{\Gamma, \neg\phi \Rightarrow \Delta}$$

also $(\vee\mathrm{L})$, $(\vee\mathrm{R})$, $(\exists\mathrm{L})$, $(\exists\mathrm{R})$, $(\rightarrow\mathrm{L})$, $(\rightarrow\mathrm{R})$

Rules for truth

Formulate the truth axioms as rules of inference.

$$(\wedge_{\mathrm{T}}\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, \mathrm{T}s_0 \qquad \Gamma \Rightarrow \Delta, \mathrm{T}s_1}{\Gamma \Rightarrow \Delta, \mathrm{T}(s_0 \dot{\wedge} s_1)} \qquad (\wedge_{\mathrm{T}}\mathrm{L})\ \frac{\Gamma, \mathrm{T}s_i \Rightarrow \Delta}{\Gamma, \mathrm{T}(s_0 \dot{\wedge} s_1) \Rightarrow \Delta}$$

$$(\forall_{\mathrm{T}}\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, \mathrm{T}\ulcorner \phi(\bar{n}) \urcorner \text{ every } n < \omega}{\Gamma \Rightarrow \Delta, \mathrm{T}\ulcorner \forall x \phi \urcorner} \qquad (\forall_{\mathrm{T}}\mathrm{L})\ \frac{\Gamma, \mathrm{T}\ulcorner \phi(s) \urcorner \Rightarrow \Delta}{\Gamma, \mathrm{T}\ulcorner \forall x \phi \urcorner \Rightarrow \Delta}$$

$$(\neg_{\mathrm{T}}\mathrm{R})\ \frac{\Gamma, \mathrm{T}s \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathrm{T}\dot{\neg} s} \qquad (\neg_{\mathrm{T}}\mathrm{L})\ \frac{\Gamma \Rightarrow \Delta, \mathrm{T}s}{\Gamma, \mathrm{T}\dot{\neg} s \Rightarrow \Delta}$$

$$(\mathrm{TR})\ \frac{\Gamma \Rightarrow \Delta, \mathrm{T}s}{\Gamma \Rightarrow \Delta, \mathrm{T}(\mathrm{T}s)} \qquad (\mathrm{TL})\ \frac{\Gamma, \mathrm{T}s \Rightarrow \Delta}{\Gamma, \mathrm{T}(\mathrm{T}s) \Rightarrow \Delta}$$

and similarly for $\exists$, $\vee$, $\rightarrow$ and $=$.

For a given set Th of truth rules, we want to show:

If $\Gamma \Rightarrow^{\alpha} \Delta$ is derivable in Th and $\Gamma \cup \Delta$ is truth-free,
then $\Gamma \Rightarrow^{\beta} \Delta$ is derivable in PA_{ω} for some determined β.

Partial cut elimination

Theorem

Let Th *be some set of sequent rules above. Then if* $\Gamma \Rightarrow^{\alpha} \Delta$ *is derivable in* Th*, then* $\Gamma \Rightarrow \Delta$ *has a derivation in which all cuts are on the atomic truth predicate only; its height is bounded by* 2^{α}_{n} *for some* n *where* $2^{\alpha}_{n+1} = 2^{2^{\alpha}_{n}}$ *and* $2^{\alpha}_{0} = \alpha$*.*

Question

Given $\Gamma \Rightarrow^{\alpha} \Delta$ *is there a cut-free derivation of* $\Gamma \Rightarrow^{\beta} \Delta$ *for some* β*? If so, for which* β*?*

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Cut elimination

Compositional truth

- CT_ω comprises all rules of PA_ω plus the compositional rules for truth ($(\wedge_{\mathrm{T}}\mathrm{R})$, $(\wedge_{\mathrm{T}}\mathrm{L})$, $(\forall_{\mathrm{T}}\mathrm{R})$, $(\forall_{\mathrm{T}}\mathrm{L})$ etc.)
- No rules of explicit self-reference.
- CT_ω permits full cut elimination. Define two rank functions $|.|_0$ and $|.|_1$ on $\mathcal{L}_{\mathrm{T}}$ as follows:

$$|R(t_1, \ldots, t_n)|_i = 0, \qquad |\mathrm{T}s|_i = \omega \cdot i,$$

$$|\neg\phi|_i = |Qv\phi|_i = |\phi|_i + 1, \quad |\phi \diamond \psi|_i = \max\{|\phi|_i, |\psi|_i\} + 1,$$

- P.c.e. entails every derivation can be turned into one in which cut formulæ have $|.|_1$-rank ω.
- A cut on $\mathrm{T}s$ can then be eliminated by induction on $|s^{\mathbb{N}}|_0$.

Cut elimination

Ramified truth

- Language: $\mathcal{L}_\delta = \mathcal{L} \cup \{\mathrm{T}_\xi \mid \xi < \delta\}$.
- Axioms of $\mathrm{RT}^{<\kappa}$ include the axioms of CT for each predicate T_λ with $\lambda < \kappa$, plus:

$$(\mathrm{T}_\delta\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, \mathrm{T}_\gamma s}{\Gamma \Rightarrow \Delta, \mathrm{T}_\delta \ulcorner \mathrm{T}_\gamma s \urcorner}\ \gamma < \delta \qquad (\mathrm{T}_\delta\mathrm{L})\ \frac{\Gamma, \mathrm{T}_\gamma s \Rightarrow \Delta}{\Gamma, \mathrm{T}_\delta \ulcorner \mathrm{T}_\gamma s \urcorner \Rightarrow \Delta}\ \gamma < \delta$$

Cut elimination

Ramified truth

- $\mathrm{RT}_{\omega}^{<\kappa}$ also permits cut elimination.
- Replace all the truth rules by axiom $\Gamma, \mathrm{T}_\lambda s \Rightarrow \Delta, \mathrm{T}_\lambda s$ and

$$(\mathrm{T}_\lambda^* \mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \mathrm{T}_\lambda \phi}\ lvl(\phi) < \lambda \qquad (\mathrm{T}_\lambda^* \mathrm{L})\ \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \mathrm{T}_\lambda \phi \Rightarrow \Delta}\ lvl(\phi) < \lambda$$

  for $\lambda < \kappa$.
- Define rank functions $|.|$ by:

$$|R(t_1, \ldots, t_n)| = 0, \qquad |\mathrm{T}_\gamma s| = \omega \cdot \gamma,$$
$$|\neg\phi| = |Qv\phi| = |\phi| + 1, \quad |\phi \diamond \psi| = \max\{|\phi|, |\psi|\} + 1,$$

- If $lvl(\phi) < \lambda < \kappa$ then $|\phi| < \omega \cdot \delta + \omega \leq |\mathrm{T}_\lambda \phi|$.

Formalised model constructions

Friedman-Sheard theory of truth

- FS extends CT by finitely iterated truth:

$$\text{Nec}\ \frac{\phi}{\mathrm{T}\ulcorner\phi\urcorner} \qquad\qquad \text{co-Nec}\ \frac{\mathrm{T}\ulcorner\phi\urcorner}{\phi}$$

- To form FS_ω, add these rule to CT_ω:

$$(\mathrm{Nec}_m)\ \frac{\varnothing \Rightarrow_m \phi}{\Gamma \Rightarrow_{m+1} \Delta, \mathrm{T}s}\ \text{if } s^{\mathbb{N}} = \ulcorner\phi\urcorner \qquad (\text{co-Nec}_m)\ \frac{\varnothing \Rightarrow_m \mathrm{T}\ulcorner\phi\urcorner}{\Gamma \Rightarrow_m \Delta, \phi}\ \text{if } s^{\mathbb{N}} = \ulcorner\phi\urcorner$$

- Adding co-Nec ruins partial cut elimination.

Formalised model constructions

Theorem

Suppose $\Gamma \Rightarrow^{\alpha}_{m} \Delta$ is derivable in FS_ω and $\Gamma \cup \Delta$ is truth-free. Then $\Gamma \Rightarrow \Delta$ has a derivation in PA_ω with height bounded by ϵ^{m}_{α} for some m, where $\epsilon^{0}_{\alpha} = \alpha$ and $\epsilon^{m+1}_{\alpha} = \epsilon_{\epsilon^{m}}$.

Proof.

- Define $\mathfrak{M}^{X}_{0} = \langle \mathbb{N}, X \rangle$ and $\mathfrak{M}^{X}_{n+1} = \langle \mathbb{N}, \{\ulcorner \phi \urcorner \mid \mathfrak{M}^{X}_{n} \models \phi\} \rangle$.
- Notice that (*) $\Gamma \Rightarrow^{\alpha}_{m} \Delta$ implies $\mathfrak{M}^{X}_{m} \models \bigwedge \Gamma \to \bigvee \Delta$ for every $X \subseteq \omega$.
- The above proof can be formalised in systems in which the operation $X \mapsto \mathfrak{M}^{X}_{0}$ is definable.
- $\{\ulcorner \phi \urcorner \mid \mathfrak{M}^{X}_{0} \models \phi\}$ is recursive in $X^{(\omega)}$, so (*) is provable in:
  - $\mathrm{ACA} + \forall X \exists Y (Y = X^{(\omega)})$ in general; or
  - $\mathrm{ACA}_0 + \forall X \exists Y (Y = X^{(\omega)})$ for "meta" m.
- $\mathrm{ACA}_0 + \forall X \exists Y (Y = X^{(\omega)})$ conservatively extends $\mathrm{PA} + \{\mathrm{TI}(<\epsilon^{m}_{0}) \mid m < \omega\}$. $\square$

Asymmetric interpretations

Kripke-Feferman truth

- KF_ω extends $\mathrm{CT}_\omega - \{(\neg_{\mathrm{T}}\mathrm{R}), (\neg_{\mathrm{T}}\mathrm{L})\}$ by strong self-applicability:

$$(\mathrm{T}_1\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, \mathrm{T}r}{\Gamma \Rightarrow \Delta, \mathrm{T}\mathrm{T}r} \qquad (\mathrm{T}_1\mathrm{L})\ \frac{\Gamma, \mathrm{T}r \Rightarrow \Delta}{\Gamma, \mathrm{T}\mathrm{T}r \Rightarrow \Delta}$$

$$(\mathrm{T}_2\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, \mathrm{T}\underset{\cdot}{\neg}r}{\Gamma \Rightarrow \Delta, \mathrm{T}\underset{\cdot}{\neg}\mathrm{T}r} \qquad (\mathrm{T}_2\mathrm{L})\ \frac{\Gamma, \mathrm{T}\underset{\cdot}{\neg}r \Rightarrow \Delta}{\Gamma, \mathrm{T}\underset{\cdot}{\neg}\mathrm{T}r \Rightarrow \Delta}$$

- KF_ω permits partial cut elimination.
- There is no canonical measure of complexity that covers both $(\mathrm{T}_1\mathrm{R})$ and $(\mathrm{T}_1\mathrm{L})$.

Asymmetric interpretations

- Define, by transfinite induction, sets S^+_α, S^-_α:

$$S^+_0 = S^-_0 = \varnothing,$$
$$S^+_{\alpha+1} = \{\ulcorner\phi\urcorner \mid \langle \mathbb{N}, S^+_\alpha, S^-_\alpha\rangle \models_{\mathrm{SK}} \phi\},$$
$$S^-_{\alpha+1} = \{\ulcorner\phi\urcorner \mid \langle \mathbb{N}, S^+_\alpha, S^-_\alpha\rangle \models_{\mathrm{SK}} \overline{\phi}\},$$
$$S^+_\lambda = \bigcup_{\alpha<\lambda} S^+_\alpha, \qquad S^-_\lambda = \bigcup_{\alpha<\lambda} S^-_\alpha,$$

- For ϕ in cnf, define $\phi^{(\alpha,\beta)}$:

$$(\mathrm{T}s)^{(\alpha,\beta)} \text{ iff } s^{\mathbb{N}} \in S^+_\alpha \qquad (\neg\mathrm{T}s)^{(\alpha,\beta)} \text{ iff } s^{\mathbb{N}} \notin S^-_\beta$$
$$(\phi \wedge \psi)^{(\alpha,\beta)} \text{ iff } \phi^{(\alpha,\beta)} \text{ and } \psi^{(\alpha,\beta)} \qquad (\forall x\phi)^{(\alpha,\beta)} \text{ iff for all } n,\ \phi(\bar{n})^{(\alpha,\beta)}$$

Lemma (Cantini)

If $\Gamma \Rightarrow^\alpha \Delta$ is derivable in KF_ω then for every β, $(\bigvee\overline{\Gamma} \vee \bigvee\Delta)^{(\beta+\omega^\alpha,\beta)}$ is true.

Outline

Axiomatisable concepts

Truth-theoretic principles can axiomatise other meta-theoretic concepts.

- Provability:
  - $\mathrm{P}\phi \wedge \mathrm{P}(\phi \to \psi) \to \mathrm{P}\psi$,
  - from ϕ infer $\mathrm{P}\phi$,
  - $\mathrm{P}\phi \to \mathrm{PP}\phi$.
- Knowledge:
  - $\mathrm{K}\phi \wedge \mathrm{K}(\phi \to \psi) \to \mathrm{K}\psi$,
  - from ϕ infer $\mathrm{K}\phi$,
  - $\mathrm{K}\phi \to \phi$.
- Intuitionistic truth:
  - Compositional axioms for $\to$, $\wedge$, $\bot$, $\forall$, etc.
  - $\mathrm{T}(\phi \vee \psi) \leftrightarrow \mathrm{T}\phi \vee \mathrm{T}\psi$ and $\mathrm{T}(\exists x\phi) \leftrightarrow \exists x\mathrm{T}(\phi(\dot{x}))$,
  - Some self-reference?
- Provability in ω-logic with reflection:
  - Axioms of provability,
  - $\forall x\mathrm{P}(\phi\dot{x}) \to \mathrm{P}(\forall x\phi)$,
  - from $\mathrm{P}\phi$ infer ϕ, or $\mathrm{PP}\phi \to \mathrm{P}\phi$, or $\mathrm{P}\phi \to \phi$.

A theory of provability with weak reflection

Consider the following intuitionistic theory S extending $\mathrm{HA} + \mathrm{Ind}_{\mathcal{L}_{\mathrm{P}}}$:

1. $\phi(x) \to \mathrm{T}\phi(\dot{x})$ for ϕ in Δ_0
2. $\forall\phi\forall\psi(\mathrm{P}\phi \wedge \mathrm{P}(\phi \to \psi) \to \mathrm{P}\psi)$
3. Compositional axioms for $\forall$, $\exists$, $\vee$ and $\wedge$:
$$\forall x\mathrm{P}(\phi\dot{x}) \leftrightarrow \mathrm{P}(\forall x\phi) \qquad \mathrm{P}(\phi \vee \psi) \leftrightarrow \mathrm{P}\phi \vee \mathrm{P}\psi$$
$$\exists x\mathrm{P}(\phi(\dot{x})) \leftrightarrow \mathrm{P}(\exists x\phi) \qquad \mathrm{P}(\phi \wedge \psi) \leftrightarrow \mathrm{P}\phi \wedge \mathrm{P}\psi$$
4. Internal reflection $\forall\phi(\mathrm{PP}\phi \to \mathrm{P}\phi)$
5. 'from ϕ infer $\mathrm{P}\phi$' and 'from $\mathrm{P}\phi$ infer ϕ'

A theory of provability with weak reflection

Outline of a proof-theoretic analysis

Steps in the proof-theoretic analysis of S:

1. Present S as a sequent calculus
2. Informal model construction
3. Formalise the consistency proof:
   - Asymmetric interpretation
   - Interpreting derivability as arithmetical truth
4. Embed theorems of S into a theory of intuitionistic truth
5. Perform cut elimination

The informal model construction

S as a sequent calculus:

$(\wedge_{\mathrm{P}}\mathrm{R})$, $(\wedge_{\mathrm{P}}\mathrm{L})$, $(\vee_{\mathrm{P}}\mathrm{R})$, $(\vee_{\mathrm{P}}\mathrm{L})$ $\qquad$ $(\forall_{\mathrm{P}}\mathrm{R})$, $(\forall_{\mathrm{P}}\mathrm{L})$, $(\exists_{\mathrm{P}}\mathrm{R})$, $(\exists_{\mathrm{P}}\mathrm{L})$

$$(\to_{\mathrm{P}}\mathrm{L})\ \frac{\Gamma \Rightarrow_m \mathrm{P}\phi \qquad \Gamma, \mathrm{P}\psi \Rightarrow_m \chi}{\Gamma, \mathrm{P}(\phi \to \psi) \Rightarrow_m \chi}$$

$$(\mathrm{PL})\ \frac{\Gamma, \mathrm{P}s \Rightarrow_m \phi}{\Gamma, \mathrm{P}\mathrm{P}s \Rightarrow_m \phi} \qquad (\mathrm{Nec})\ \frac{\varnothing \Rightarrow_m \phi}{\Gamma \Rightarrow_n \mathrm{P}\ulcorner\phi\urcorner}\ m < n$$

Then

- $\Rightarrow_{m+1}$ has the disjunction and existence property, and
- If $\Gamma \Rightarrow_{m+1} \phi$ is derivable then $\langle \mathbb{N}, \{\ulcorner\psi\urcorner \mid \varnothing \Rightarrow_m \psi\}\rangle \models (\bigwedge \Gamma) \to \phi$.
- $\varnothing \Rightarrow_{m+1} \mathrm{P}\phi$ implies $\varnothing \Rightarrow_m \phi$.

So if $\mathrm{S} \vdash \phi$ then $\varnothing \Rightarrow_m \phi$ for some m.

In which system can the soundness proof be formalised?

Formalising the model construction

Asymmetric interpretations

We make use of asymmetric interpretations:

For $m < \omega$, α and β, define $A^{(\alpha,\beta,m)}$ by:

$$
\begin{aligned}
(\mathrm{P}\phi)^{(\alpha,\beta,m)} &\text{ iff } \exists\gamma < \varphi_m\alpha\exists n < m(\varnothing \Rightarrow_n^\gamma \phi) \\
(A \to B)^{(\alpha,\beta,m)} &\text{ iff } (A^{(\beta,\alpha,m)} \to B^{(\alpha,\beta,m)}) \wedge \exists\gamma < \varphi_m\alpha(A \Rightarrow_m^\gamma B) \\
(A \circ B)^{(\alpha,\beta,m)} &\text{ iff } (A^{(\alpha,\beta,m)} \circ B^{(\alpha,\beta,m)}) \quad (\circ \in \{\wedge, \vee\}) \\
(QxA)^{(\alpha,\beta,m)} &\text{ iff } (Qn\, A(n)^{(\alpha,\beta,m)}) \quad Q \in \{\forall, \exists\}
\end{aligned}
$$

Lemma

If $\Gamma \Rightarrow_m^\alpha \phi$ then for every β, $\mathbb{N} \models (\bigwedge \Gamma \to \phi)^{(\beta+\omega^\alpha,\beta,m)}$.

Question

How do we formalise $\mathbb{N} \models \phi^{(\alpha,\beta,m)}$?

Formalising the model construction

Formalising truth

Question

How do we formalise $\mathbb{N} \models \phi^{(\alpha,\beta,m)}$*?*

Answer

Use a truth predicate:

$\Gamma \Rightarrow^{\alpha}_{m} \phi$ *implies* $\mathrm{CT}^{i} + \mathrm{TI}(<\varphi_{m+1}\alpha) \vdash \forall\beta\ \mathrm{T}(\ulcorner(\bigwedge\Gamma \to \phi)^{(\omega^{\alpha},\beta,m)}\urcorner)$.

Thus, if $\Gamma \Rightarrow^{\alpha}_{m} \phi$ is derivable in S_{ω} and Γ, ϕ are truth free,

1. $\varnothing \Rightarrow^{\beta} \mathrm{T}(\bigwedge\Gamma \to \phi)$ is derivable in CT^{i}_{ω} for some $\beta < \varphi_{\omega}0$,
2. $\Gamma \Rightarrow^{\beta} \phi$ is derivable in CT^{i}_{ω} for some $\beta < \varphi_{\omega}0$,
3. $\Gamma \Rightarrow \phi$ is cut-free derivable in CT^{i}_{ω} with height $<\varphi_{\omega}0$,
4. $\Gamma \Rightarrow \phi$ is derivable in $\mathrm{HA} + \mathrm{TI}(<\varphi_{\omega}0)$.

So S is a conservative extension of $\mathrm{HA} + \mathrm{TI}(<\varphi_{\omega}0)$. $\square$