Semantics in Type Theory

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Overview

Two logical traditions

Type theory and early metalogic

Methods of domain variation

1. Type relativization
2. Type flexibilization

Summary

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The (pre-)history of model theory

A standard picture:

- A distinction between two logical traditions: the ‘algebraic’ (Schröder, Löwenheim, Skolem) and the ‘type-theoretic’ tradition (Russell, Carnap, Tarski, Quine...)

(Vaught (1974); Badesa (2004); Goldfarb (1979), Hintikka (1988), Haaparanta (2009), etc.)

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- Development of model-theoretic semantics is usually associated with the algebraic tradition ...
- ... while the type-theoretic tradition is considered to be largely independent from the "birth of model theory".

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A central philosophical assumption underlying this historical picture:

- The two traditions are based on two incompatible *conceptions* of logic, namely the "universalist" and the "model-theoretic" conception.
- Moreover, for proponents of the universalist view–such as Frege, Russell, as well as others working with *Principia Mathematica*-style logics—model theory and metatheory were principally inconceivable.

Logical universalism vs. model theory

As a result, Frege's and Russell's systems are meant to provide a universal language (...). There can be no position outside the system from which to assess it. (...) questions of disinterpretation or reinterpretation cannot arise. (...) Frege and Russell can have no notion of "interpretation," or of "semantics." (...) Moreover, the logic they practice aims only at issuing general truths in this language, In particular, it does not issue meta-statements of the form "X is a logical truth" or "X implies Y. (Goldfarb 1983, 694)

Logical universalism vs. model theory

According to the tenents of this tradition, you are a prisoner (as it were) of your language. You cannot step outside it, you cannot re-interpret it in a large scale, and you cannot even express its semantics in the language itself. (Hintikka 1988, 1)

These views were in the main shared by early Russell, Wittgenstein, in the Vienna Circle in its hayday in 1930-32, Quine, etc. (ibid, 2)

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- Show that important advances to model-theoretic semantics were made in the type-theoretic tradition.
- In particular, several attempts to express the model theory of axiomatic theories *within* a type theoretic framework prior to the turn to modern metatheory.

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Logic in the 1930s: a ‘zenith’ of type theory

- Ramified TT introduced in *Principia Mathematica* (Whitehead & Russell 1910-1913) for the logicist reconstruction of mathematics;

(see Ferreiros (2001), Grattan-Guinness (2000))

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- ... used not as a "foundational system for mathematics" but as the "natural" logical system (prior to the consolidation of FOL)
- ... several non-foundationalist applications: philosophy (Carnap's *Logischer Aufbau*), axiomatics, metamathematics, etc.

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The language of STT

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The *type* of an expression is defined inductively:

(i) i is the type of individual expressions;

(ii) the type of an n-ary relation $R(t_1, \ldots, t_n)$ with arguments of types $\alpha_1, \ldots, \alpha_n$ is $\langle \alpha_1 \ldots \alpha_n \rangle$.

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- $\{D_\alpha\}_{\alpha \in \tau}$ is a *frame* of type domains:
  - (i) D_i is a set of individuals;
  - (ii) for any type α: $D_{\langle \alpha \rangle} = \wp(D_\alpha)$;
  - (iii) for any type $\langle \alpha_1 \ldots \alpha_n \rangle$, $D_{\langle \alpha_1 \ldots \alpha_n \rangle} = \wp(D_{\alpha_1} \times \cdots \times D_{\alpha_n})$.

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- I is an interpretation function that assigns to each constant expression of type α an element from D_α.

Formal axiomatics in STT

STT used for the formalization of axiomatic systems and their metatheory:

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- 'Metatheory' for axiomatic theories in Carnap's manuscript *Untersuchungen zur allgemeinen Axiomatik* (Carnap 2000)
- Tarski's work on the "methodology of the deductive sciences" (e.g. Tarski 1930a, 1930b, Tarski & Lindenbaum 1934, Tarski 1935)

Formalization in STT

Logical formalization of "formal" axiomatic theories:

- Primitive terms of a theory expressed as variables (of given arity and type) $X_1, \ldots, X_n$ (and not, as is usual today, in terms of schematic nonlogical constants).

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- Axioms, axiom systems and theorems expressed as sentential functions, i.e. as formulas $\Phi(\vec{X})$ in modern sense.
- Models $\mathfrak{M}$ as n-tuples of relations $\langle \mathrm{R}_1, \ldots, \mathrm{R}_n \rangle$ "living" in $\mathcal{V}$ where each R_i is assigned to the "primitive signs" X_i in Φ.

Carnap on models

If fR is satisfied by the constant R_1, where R_1 is an abbreviation of a system of relations $P_1, Q_1, \ldots$; then R_1 is called a "model" of f. A model is a system of concepts of the basic system, generally a system of numbers (number classes, relations and so forth). (Carnap 1930, 303)

An example

A monoid is a triple $\langle G, \circ, e \rangle$, where G is a set, $e \in G$, and $\circ$ is a binary operation $G \times G \to G$ that satisfies two axioms:

$$\textbf{1.}\ \forall x, y, z \in G : (x \circ y) \circ z = x \circ (y \circ z)$$
$$\textbf{2.}\ \forall x \in G : e \circ x = x \circ e = x$$

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A type-theoretic formalization contains a unary relation variable $X(x)$ of type $\langle i\rangle$, a binary function variable $f(x, y)$ of type $\langle i, i : i\rangle$, and an individual variable v of type i.

$$\begin{aligned}\Phi(X, f, v) =_{df} \forall x \forall y \forall z (X(x) \wedge X(y) \wedge X(z) \to \\ (f(f(x, y), z) = f(x, f(y, z)) \wedge \\ f(v, x) = f(x, v) = v))\end{aligned}$$

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A model $\mathfrak{M}$ of $\Phi(X, f, v)$ is a triple $\langle R, \mathbf{f}, a\rangle$ of type $\langle\langle i\rangle, \langle i, i : i\rangle, i\rangle$ that satisfies the theory.

Metatheoretic notions

In Carnap's *Untersuchungen zur allgemeinen Axiomatik* (1928), the following metatheoretic notions introduced, e.g.:

- Logical consequence / "*Lehrsätze*":

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(See Awodey & Carus 2001, Awodey & Reck 2002, Reck 2007)

... similar notions in Tarski's work

- Tarski (1935a): for a theory $\psi(a, b, c, \dots)$, categoricity expressed as:

$$(x', x'', y', y'', z', z'', \dots) : \psi(x', y', z', \dots).$$
$$\psi(x'', y'', z'', \dots). \supset .(\exists \mathrm{R})\mathrm{R}\frac{x', y', z', \dots}{x'', y'', z'', \dots}$$

- in Tarski (1940): A theory $P(C)$ is categorical 'relative to its logical basis':

$$(X)(Y)[\mathrm{P}(X)\&\mathrm{P}(X) \to X \sim Y]$$

- The logical basis in both is "the logic of *Principia Mathematica*."

(see Mancosu (2010) for details.)

STT vs. modern metatheory

Differences to the modern approach:

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- The truth of 'metatheoretic' statements about a theory is specified in terms of informal notion of *truth in STT*.

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- The truth of 'metatheoretic' statements about a theory is specified in terms of informal notion of *truth in STT*.
- Thus, the fact that φ *follows* from Γ—i.e. the modern $(\forall\mathfrak{M})(\mathfrak{M} \models \Gamma \Rightarrow \mathfrak{M} \models \varphi)$—seems to be understood as:

$$\mathcal{V} \models \forall X(\Gamma(X) \to \varphi(X))$$

The metatheoretic turn

A shift to metatheoretic semantics in Carnap and Tarski's work in the 1930s:

- Tarski, *The concept of truth in formalized languages*, 1935
- Carnap, *Logische Syntax der Sprache*, 1934
- ... metatheoretic definitions of satisfaction, truth, and analyticity *for* type-theoretic languages.

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However, one continuity with earlier work:

⇒ STT *now as an object language* still not conceived as a formal language in modern sense but as a "meaningful formalism".

Analyticity in LII

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- $Analytic_{\mathrm{LII}}(\varphi)$ is a metatheoretic explication of "truth in STT". (A close variant of Tarski's notion of truth in Tarski (1933)).
- A definition of analyticity in terms of a definition of valuations and a binary predicate $A(\varphi, \alpha)$—"analytic in respect of a certain valuation":

$$A(\varphi, \alpha) \Leftrightarrow_{df} \mathcal{V} \models \varphi[\alpha]$$

Analyticity *in respect of valuation* α

A "modernized" definition of $A(\varphi, \alpha)$:

1. $A(t_1^i = t_2^i, \alpha) \Leftrightarrow \alpha(t_1^i) = \alpha(t_1^i)$
2. $A(R(t_1^i, \ldots, t_n^i), \alpha) \Leftrightarrow \langle \alpha(t_1^i), \ldots, \alpha(t_n^i) \rangle \in \alpha(R)$
3. $A(\neg\psi, \alpha) \Leftrightarrow \neg A(\psi, \alpha)$
4. $A(\psi \wedge \chi, \alpha) \Leftrightarrow A(\psi, \alpha) \wedge A(\chi, \alpha)$
5. $A(\forall v \psi, \alpha) \Leftrightarrow \forall \beta A(\psi, \beta)$,
   where β differs from α at v at most.

Carnap (1934) on analyticity in LII:

$$Analytic_{\mathrm{LII}}(\varphi) \Leftrightarrow \forall \alpha A(\varphi, \alpha)$$

Analyticity and models of theories

Carnap & Bachmann (1936) on *truth in a model* for axiomatic theories:

> *Let 'M_1' be an abbreviation for a sequence of constants of the language S. We say that M_1 is a model of the axiom system '$F_1(M)$' if the sentence '$F_1(M_1)$' is analytic in S [i.e. LII]. (ibid, p.67)*

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What Carnap must mean here is this:

Definition:
Theory $\Phi(\vec{X})$ is true in a model $\mathfrak{M} = \langle \vec{R} \rangle$ iff

$$A(\Phi(\vec{X}), \alpha_{\mathfrak{M}}) \text{ or } \mathcal{V} \models \Phi(\vec{X})[\alpha_{\mathfrak{M}}]$$

where $\alpha_{\mathfrak{M}}(X_i) = R_i$.

A difference to the modern account

Given Carnap's notion of analyticity, the "metatheoretic" quantification over models in concepts like *logical consequence* can now be expressed in terms of quantification over valuations.

E.g. "*satisfiability*" $\exists X(\Phi(\vec{X}))$ becomes $\exists \alpha A(\Phi(\vec{X}), \alpha)$.

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But in the particular case of *truth in a model*, $A(\Phi, \alpha_{\mathfrak{M}})$, how are the "object-theoretic" quantifiers in Φ made to be about the domain of $\mathfrak{M}$? Recall that variables of $\mathcal{L}_{STT}$ are stipulated to range over fixed type domains in $\mathcal{V}$.

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A central interpretive question: How is this interpretation in a given model domain and thus domain variation (for axiomatic theories) simulated in a fully interpreted type theory?

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... but also in terms of *restricting* the object-theoretic quantifiers in a fully *interpreted* language to different subdomains via *relativization*.

Version (1): Relativization via “domain predicates”

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The type universe is thus *relativized* to the extension of $P(x)$ under a given interpretation of the theory (in the above sense).

The use of domain predicates

The trick consists in relativizing the quantifier to a predicate for a domain $D(x)$ and then one can, within the same universal domain, obtain all the possible interpretations of $D(x)$. In addition, fixed-domain conceptions of consequence were not rare at the time of Tarski's writing and were in fact upheld by logicians with logicist leanings (Carnap, Ramsey, Russell, Lewis, Langford, etc.) who worked with type theories with a fixed domain of individuals, given at the outset (...). (Mancosu 2010)

An example

A type-theoretic formalization of the theory of monoids:

$$\Phi(X, f, v) =_{df} \forall x \forall y \forall z (X(x) \wedge X(y) \wedge X(z) \rightarrow (f(f(x, y), z) = f(x, f(y, z)) \wedge (ii) f(v, x) = f(x, v) = v))$$

A model $\mathfrak{M}$ of $\Phi(X, f, v)$ is a triple $\langle R, \mathbf{f}, a \rangle$ of type $\langle \langle i \rangle, \langle i, i : i \rangle, i \rangle$ that satisfies the theory.

The equivalence with modern model-theoretic truth

(1) The modern approach:
An informal theory T (with signature $\{R_1, \ldots, R_n\}$) is expressed as a (set of) sentences Φ in language $\mathcal{L}$. A model $\mathfrak{M}$ of T is a tuple $\langle Dom(\mathfrak{M}), I\rangle$. Truth in a model is expressed as $\mathfrak{M} \models \Phi$.

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(2) The type-theoretic approach:
T is formalized in $\mathcal{L}_{STT}$ as a formula Φ' with free variables $X, Y_1 \ldots, Y_n$. Φ contains a unary "domain" variable X of type $\langle i\rangle$ such that, for any interpretation $\mathfrak{M}$ of T, (i) $Dom(\mathfrak{M}) \subseteq D_0$ and (ii) $Dom(\mathfrak{M}) = \alpha_{\mathfrak{M}}(X)$

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(1) and (2) are equivalent:
For any model, we have: $\mathfrak{M} \models \Phi \Leftrightarrow \mathcal{V} \models \Phi'[\alpha_{\mathfrak{M}}]$

Relativization Theorem

Theorem (Relativization Theorem)

For every formula $\phi \in \mathcal{L}$ there exists a formula $\phi^P \in \mathcal{L}^+$ such that if $\mathfrak{N}_P$ is a P-part of $\mathfrak{M}$ and $\bar{a}$ a sequence of elements from $Dom(\mathfrak{N})$, then

$$\mathfrak{N}_P \models \phi[\bar{a}] \Leftrightarrow \mathfrak{M} \models \phi^P[\bar{a}]$$

(Hodges 1997, 203)

Version (2): relativization without “domain predicates”

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1. How are model domains conceived by him (in particular, in axiomatizations without a domain predicate)?
2. How is the type relativization to model domains effected without the explicit use of domain predicates?

A “domain-as-fields” conception of models

- A formal model of Φ is a relational structure $\mathfrak{M} = \langle \mathrm{R}_1, \ldots, \mathrm{R}_n \rangle$, where, for simplicity, we assume that each n-ary $R_i \subseteq D_0^n$.

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- Models of theories are substructures of $\mathcal{V}$, model domains subsets of D_0.

Implicit semantic relativization?

Instead of explicit relativization via domain predicates, a *holistic* semantic relativization in terms of implicitly relativizing the range of quantifiers to the extension of the primitive terms of a theory.

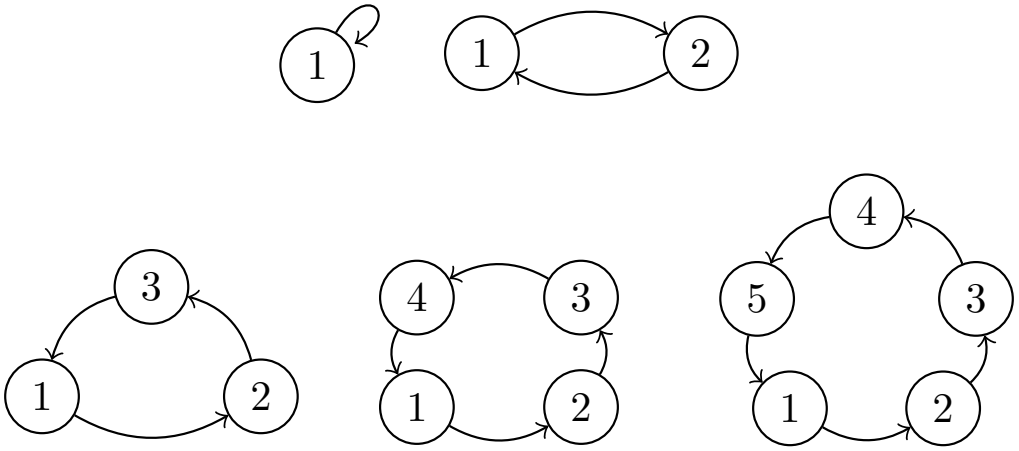
Thus, any "model assignment" $\alpha_{\mathfrak{M}}$ to $\Phi(\vec{X})$ should implicitly relativize the quantifiers in Φ to $Dom(\mathfrak{M}) \subseteq D_0$.

An example: Carnap on basic arithmetic

An axiom system of basic arithmetic with one primitive term $R(x, y)$:

1. $\forall x \forall y (R(x, y) \rightarrow \exists z (R(y, z)))$
2. $\forall x \forall y \forall z ((R(x, y) \wedge R(x, z) \rightarrow y = z) \wedge (R(x, y) \wedge R(z, y) \rightarrow x = z))$
3. $\exists ! x (x \in Dom(R) \wedge x \notin Ran(R))$
4. Minimal axiom (comparable to the induction axiom)

Models of subtheory A1 & A2:



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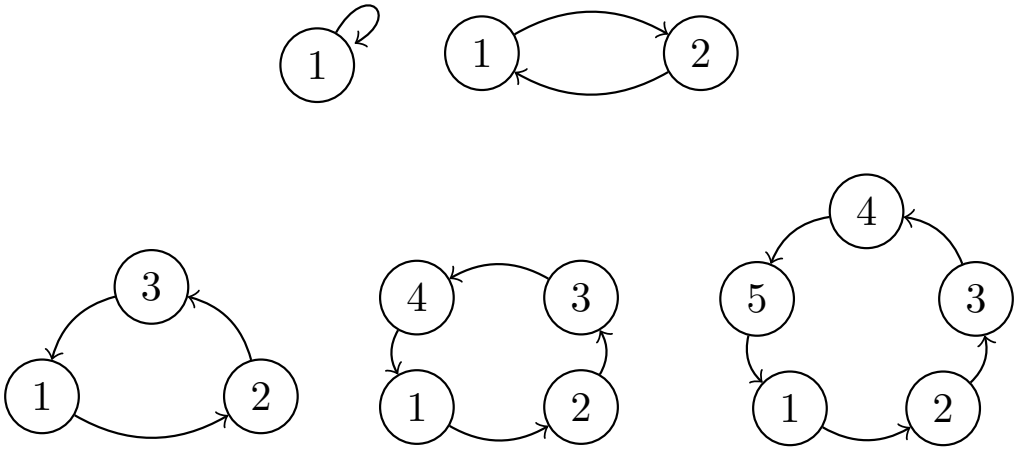

Note that any model $\mathfrak{M}_C$ of theory $A1\&A2$ consists only of a relation $\mathbf{R} \subseteq D_0 \times D_0$ that is assigned to the primitive variable $R(x, y)$. Moreover, every "interpretation" of $R(x, y)$ effectively restricts to quantifiers in A1 & A2 to a *different* model domain $D_{\mathfrak{M}} = Fld(\mathbf{R}) \subseteq D^0$.

The use of domain predicates in *Abriss*

Carnap's *Abriss der Logistik* (1929):

- Several axiomatic theories expressed in STT in Part II (titled "applied logistic"):
- Here, Carnap explicitly uses domain predicates (or domain variables) to restrict type domains to particular model domains.
- A conjecture: domain predicates are added to the primitive vocabulary of theory by Carnap *precisely* in cases where an implicit relativization would fail!

Axiomatics in *Abriss* (1)

Peano arithmetic with primitive signs $\{nu, Za, Nf\}$:

Ax 2 $\forall x(x \in za \rightarrow Nf(x) \in za)$

Ax 3 $\forall x \forall y(x, y \in za \wedge Nf(x) = Nf(y) \rightarrow x = y)$

Ax 4 $\forall x(x \in za \rightarrow Nf(x) \neq 0)$

Axiomatics in *Abriss* (2)

Topology (*Hausdorff Axioms*) (Primitive signs: $\{\alpha \mathbf{U} x\}$, (α is a neighborhood set of x):

Def Class of points defined as: $\mathbf{pu} := Ran(\mathbf{U})$

A 1 $Dom(\mathbf{U}) \subset \wp(\mathbf{pu})$

A 2 $\forall \alpha, \beta, x(\alpha \mathbf{U} x \wedge \beta \mathbf{U} x \rightarrow \exists \gamma(\gamma \mathbf{U} x \wedge \gamma \subset \alpha \cap \beta))$

A 3 $\forall \alpha, y(\alpha \in Dom(\mathbf{U}) \wedge y \in \alpha \rightarrow \exists \gamma(\gamma \mathbf{U} y \wedge \gamma \subset \alpha))$

A 4 $\forall x, y(x, y \in \mathbf{pu} \wedge x \neq y \rightarrow \exists \alpha, \beta(\alpha \mathbf{U} x \wedge \beta \mathbf{U} y \wedge \alpha \cap \beta = \emptyset))$

Axiomatics in *Abriss* (3)

Projective Geometry (one primitive sign: $\{\mathbf{ger}\}$, the class of straight lines):

Def Set of points **pu** is defined as the union of fields of elements in **ger**: $\mathbf{pu} := \bigcup_{G \in \mathbf{ger}} Fld(G)$

Ax 3 $\forall x, y(x, y \in \mathbf{pu} \wedge x \neq y \rightarrow \exists! G(x, y))$

Ax 5 $\forall x, y \exists! G(x, y \in G \rightarrow \exists z(z \in \mathbf{pu} \wedge z \notin G))$

Two logical traditions

Type theory and early metalogic

Methods of domain variation

1. Type relativization
2. Type flexibilization

Summary

Type flexibilization

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Generally, these are different ways of reconceptualizing the relation between the semantics and syntax of STT.

From typical ambiguity to flexible types

1. In Carnap's work on general axiomatics, a higher-order language without "a rigid type structure" is outlined whose variables "have no definite type but run through a denumerably infinite sequence of types" (Carnap and Bachmann 1936, 85).

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2. Similar suggestions on the abstraction from fixed type assignments can be found in Tarski, for instance in his work on formal definability (Tarski 1931).
3. Type flexibilization was investigated most extensively by Quine in *A system of logicistic* (Quine 1934). A "liberalization of the theory of types" (Church) resulting from the omission of type specifications of the expressions of the language.

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The idea is elaborated in detail in the second unpublished part of his manuscript *Untersuchungen zur Allgemeinen Axiomatik* from 1928 (RC 081-01-01 to 081-01-33).

Domain variation via flexible types

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> *There is notably as difference between M and f: M consists of constants, thus any two constituents of M therefore have a specific type-relation, they are either of the same or of a different type and if different, then in a certain relation to each other; in contrast, k_1 and k_2 are variables, there are 3 possibilities: of the same type, of different types, undetermined. (RC 081-01-16)*

Domain variation via type ascent

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> *The type of the individual domains does not at all have to be the same (since the AS can stipulate for 2 such domains either that they are type-identical or it lets the type relation undetermined (...)). (RC 081-01-18)*

Non-rigid types in 1936

The same idea is also outlined in Carnap & Bachmann (1936), again in the context of axiomatics, but this time specified more explicitly as an alternative to standard rigid understanding of STT.

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As a consequence, a "type-flexible presentation" of theories in STT necessary to reconstruct these kinds of domain extensions.

Non-rigid types in 1936

The cornerstone here is a flexible treatment of type ascriptions to the variables:

> *The variables (...) would be flexible as to type, i.e., they would have no definite type but run through a denumerably infinite sequence of types, beginnings with the "base type" of the given sign. (ibid, 85)*

A type-flexible presentation of theories

The central effect of using non-rigid type theory in the formalization of theories:

> *To a given axiom system then there may belong models of different levels. (ibid, 85)*

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> *To a given axiom system then there may belong models of different levels. (ibid, 85)*

Moreover:

> *The class of admissible models of an axiom system is much more comprehensive in this language form than in the other one. (ibid, 85)*

An example

Consider again the case of the theory of monoids:

$$\Phi(X, f, v) =_{df} \forall x \forall y \forall z (X(x) \wedge X(y) \wedge X(z) \rightarrow (f(f(x,y),z) = f(x,f(y,z)) \wedge f(v,x) = f(x,v) = v))$$

Simple examples of monoids are $\langle \mathbb{Q}, \times, 1 \rangle$ and $\langle \mathbb{R}, \times, 1 \rangle$ The individual domains $\mathbb{Q}$ and $\mathbb{R}$ have different types, and can both be presented as type theoretic objects.

If the types of free variables X, f, and v are left undetermined, then both structures can be taken as models of $\Phi(X, f, v)$.

Model variation via flexible type assignments

Model variation for theory $\Phi(X, f, v)$ can be effected by assignments of "type-distinct" objects in $\mathcal{V}$ to the primitive signs X, f, v.

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Consider, for instance, two different assignments to the "basic variables" X and f:

1. $\alpha : X \mapsto D_i$ and $\alpha : f \mapsto A \subseteq D_{\langle i,i:i\rangle}$
2. $\alpha' : X \mapsto D_{\langle i\rangle}$ and $\alpha' : f \mapsto A' \subseteq D_{\langle\langle i\rangle,\langle i\rangle:\langle i\rangle\rangle}$

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Thus, the truth of the theory of monoids $\Phi(X, f, v)$ in type-theoretic representations of $\langle \mathbb{Q}, \times, 1\rangle$ and $\langle \mathbb{R}, \times, 1\rangle$ can be recast in Carnap's terms as $A(\Phi, \alpha)$ and $A(\Phi, \alpha')$ respectively.

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Summary

Summary & Outlook

- Focus in this talk was on contributions to model-theoretic semantics within the type-theoretic tradition before and after the “metatheoretic turn”.
- Methods of type relativization and type flexibilization were central methods used to simulate a model-theoretic conception within STT.

Summary & Outlook

- Focus in this talk was on contributions to model-theoretic semantics within the type-theoretic tradition before and after the "metatheoretic turn".
- Methods of type relativization and type flexibilization were central methods used to simulate a model-theoretic conception within STT.

Open questions:

- To what extent do these results relativize the van Heijenoort/Hintikka/Goldfarb-paradigm of two incompatible logical traditions?
- Can one speak of a genuine second model-theoretic tradition where these contributions in type theory play a formative role?

Thank you.