The theory of enumeration degrees and its fragments

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Abstract. We survey a line of investigation into the theory of the enumeration degrees. Enumeration reducibility captures a model of computation based on positive information. We will focus on the theory of the associated degree structure in the language of partial orders and its fragments, built by restricting the quantifier complexity of statements. We will consider the local substructure of the enumeration degrees captured by the degrees that are computationally weaker than the Halting set. We consider how things change when we change the signature of the language.

Keywords: enumeration degrees · decidability· theory· fragments

This is the extended abstract of my talk for the conference CiE 2024 "Twenty years of theoretical and practical synergies". I will present a line of investigation into one aspect of the structure of the enumeration degrees—one that asks: how complicated is the theory of the structure. This line of investigation accompanies the study of every computability theoretic reducibility. It was carried out extensively for the Turing degrees. The enumeration degrees can be viewed as an extension of the Turing degrees. We will keep track of relevant result in both structures and point out aspects in which they differ.

The partial order of the enumeration degrees arises from the relation *enumeration reducibility*, introduced by Friedberg and Rogers [3] in 1959.

Definition 1 (Friedberg and Rogers [3]). *A set A is* enumeration reducible *to a set B (denoted by $A \leq_e B$) if there is a c.e. set Φ, such that*

$$A = \Phi(B) = \{n : \exists u(\langle n, u\rangle \in \Phi \ \&\ D_u \subseteq B)\},$$

where D_u denotes the finite set with code (canonical index) u under the standard coding of finite sets.

Equivalent forms of enumeration reducibility were actually introduced by several authors independently: see Kleene [7], Myhill [13], Uspensky [24], Selman [17]. The most common motivation for their introduction was to extend the notion of Turing reducibility to partial function. Scott [16] showed that enumeration reducibility on c.e. sets gives rise to a structure that interprets untyped lambda calculus, thus we can be confident that the reducibility provides a robust notion of computation that is intrinsically interesting.

To each reducibility we associate a degree structure in which we identify sets that are reducible to each other. The structure of the Turing degrees $\mathcal{D}_T$ has a natural embedding in the structure of the enumeration degrees $\mathcal{D}_e$: we map the Turing degree of a set A to the enumeration degree of the set $A \oplus \overline{A}$. In this sense we view $\mathcal{D}_e$ as an extension of $\mathcal{D}_T$.

This embedding is non-trivial and, in fact, the two structures are not even elementary equivalent. Spector [23] proved that the Turing degrees have minimal elements, while Gutteridge [5] showed that the enumeration degrees are downwards dense.

We focus on understanding the complexity of the sets of statements in the language of partial orders that are true in the Turing degrees and that are true in the enumeration degrees. We denote these sets by $Th(\mathcal{D}_T)$ and $Th(\mathcal{D}_e)$ and call them the *theory* of the degree structure. We will see that in each case these theories are maximally complex: as complicated as the theory Second order arithmetic. These results are due to Simpson [20] for $\mathcal{D}_T$ and to Slaman and Woodin [22] for $\mathcal{D}_e$. We will describe the general method of Slaman and Woodin [21] that codes models of arithmetic inside the degree structure and also comment on an alternative proof for the complexity of $Th(\mathcal{D}_e)$ that relies on the definability of the Turing degrees inside the enumeration degrees by Cai et. al [1].

The local structure of the enumeration degrees $\mathcal{D}_e(\leq \mathbf{0}_e')$ consists of the interval degrees $[\mathbf{0}_e, \mathbf{0}_e']$. It contains as a substructure the image of the interval $[\mathbf{0}_T, \mathbf{0}_T']$, which constitutes the local structure of the Turing degree $\mathcal{D}_T(\leq \mathbf{0}_T')$. It is a countable structure and its element have simple definitions in terms of the arithmetical hierarchy: the enumeration degrees of Σ^0_2 sets. Cooper [2] proved that $\mathcal{D}_e(\leq \mathbf{0}_e')$ is dense, while Sacks [15] showed that minimal Turing degrees exist even in $\mathcal{D}_T(\leq \mathbf{0}_T')$. The theories of the local structures are also maximally complex: each is computably isomorphic to first order arithmetic [19, 4].

In each of the cases above we may decide to restrict our attention to simpler statements: we will borrow notation from the arithmetical hierarch and say that a statement of the form $(\exists x_1)(\forall x_2)\dots(Qx_n)\varphi$, where Q is the appropriate quantifier after $n-1$ alternations and φ is quantifier free, a Σ_n formula. Π_n is defined similarly when we start with $\forall$. The Σ_n-Theory of a structure $\mathcal{D}$ consists of all Σ_n statements true in $\mathcal{D}$ and is denoted by Σ_n-$Th(\mathcal{D})$.

The Σ_1 theories of $\mathcal{D}_T$, $\mathcal{D}_e$, $\mathcal{D}_T(\leq \mathbf{0}_T')$ and $\mathcal{D}_e(\leq \mathbf{0}_e')$ are each decidable. They can be reformulated as a structural question asking which finite partial orders can be embedded in the structure. In each case the answer is all. And so at the one end, when we consider only existential statements, we have decidable theories. At the other, when we allow arbitrarily long quantifier alternations, we have highly undecidable theories. Naturally, we wonder where does decidability break down.

In all cases the Σ_3-theory of the degree structure is undecidable. We will review these results due to Lerman and Schmerl (see [11]) for the local and global

Turing degrees and to Kent [6] for the local and global enumeration degrees. We will describe the Nies Transfer Method [14] that gives a general recipe on how to prove such results.

The question that remains is then what happens at level 2. In the Turing degrees both global [10, 18] and local [12] we have an algorithm to decide where a two-quantifier statement is true or not. The algorithm largely relies on a generalization of the existence of minimal degrees. In $\mathcal{D}_e$ and in $\mathcal{D}_e(\leq \mathbf{0}_e')$ these questions remain open. We will discuss partial progress made towards such a solution and the obstacles ahead. The work discussed features in [9] and [8].

Finally we will consider how things change if we change the signature of the language: what happens if we add a function symbol for the jump operator, the skip operator, or the least upper bound operator.

Acknowledgements The author is supported by NSF Grant No. DMS-2053848.

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