Admissibility of $\Pi_2$-Inference Rules: interpolation, model completion, and contact algebras

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Logic4Peace, 22 April 2022
$\Pi_2$-rules
An inference rule \( \rho \) is a \( \Pi_2 \)-rule if it is of the form

\[
\frac{F(\phi/x, y) \to \chi}{G(\phi/x) \to \chi}
\]

where \( F(x, y) \), \( G(x) \) are propositional formulas.

We say that \( \theta \) is obtained from \( \psi \) by an application of the rule \( \rho \) if

\[
\psi = F(\phi/x, y) \to \chi \quad \text{and} \quad \theta = G(\phi/x) \to \chi,
\]

where \( \phi \) is a tuple of formulas, \( \chi \) is a formula, and \( y \) is a tuple of propositional letters not occurring in \( \phi \) and \( \chi \).
**Definition**

An inference rule \( \rho \) is a \( \Pi_2 \)-rule if it is of the form

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\begin{align*}
F(\phi/x, y) \rightarrow \chi \\
\implies \\
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\end{align*}
\]

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where \( \phi \) is a tuple of formulas, \( \chi \) is a formula, and \( y \) is a tuple of propositional letters not occurring in \( \phi \) and \( \chi \).

Let \( S \) be a propositional modal system. We say that the rule \( \rho \) is **admissible** in \( S \) if \( \vdash_{S+\rho} \phi \) implies \( \vdash_S \phi \) for each formula \( \phi \).
First method

Conservative extensions
We say that $\phi(x) \land \psi(x, y)$ is a conservative extension of $\phi(x)$ in $S$ if

$$\vdash_S \phi(x) \land \psi(x, y) \rightarrow \chi(x) \text{ implies } \vdash_S \phi(x) \rightarrow \chi(x)$$

for every formula $\chi(x)$. 

Theorem

If $S$ has the interpolation property, then a $\Pi_2$-rule $\rho$ is admissible in $S$ iff $G(x) \land F(x, y)$ is a conservative extension of $G(x)$ in $S$.

Therefore, if $S$ has the interpolation property and conservativity is decidable in $S$, then $\Pi_2$-rules are effectively recognizable in $S$.

Corollary

The admissibility problem for $\Pi_2$-rules is NexpTime-complete in $K$ and $S5$; in ExpSpace and NexpTime-hard in $S4$. 
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- $\text{NexpTime}$-complete in $K$ and $S5$;
- in $\text{ExpSpace}$ and $\text{NexpTime}$-hard in $S4$. 
Second method

Uniform interpolants
An S5-modality $[\forall]$ is called a universal modality if

$$\vdash_{S} \bigwedge_{i=1}^{n} [\forall] (\phi_{i} \leftrightarrow \psi_{i}) \rightarrow ([\square] \phi_{1}, \ldots, \phi_{n} \leftrightarrow [\square] \psi_{1}, \ldots, \psi_{n})$$

for every modality $[\square]$ of $S$. 
An S5-modality $[\forall]$ is called a universal modality if

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for every modality $[\Box]$ of $S$.

If $\phi(x, y)$ is a formula, its right global uniform pre-interpolant $[\forall_x \phi(y)]$ is a formula such that for every $\psi(y, z)$ we have that

$$\psi(y, z) \vdash_S \phi(x, y) \; \text{iff} \; \psi(y, z) \vdash_S [\forall_x \phi(y)].$$
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$$\psi(y, z) \vdash_S \phi(x, y) \text{ iff } \psi(y, z) \vdash_S \forall_x \phi(y).$$

**Theorem**

Suppose that $S$ has uniform global pre-interpolants and a universal modality $[\forall]$. Then a $\Pi_2$-rule $\rho$ is admissible in $S$ iff

$$\vdash_S [\forall] \forall_y (F(x, y) \rightarrow z) \rightarrow (G(x) \rightarrow z).$$
Third method

Simple algebras and model completions
To a $\Pi_2$-rule we associate the first-order formula

$$\Pi(\rho) := \forall x, z \left( G(x) \not\leq z \Rightarrow \exists y : F(x, y) \not\leq z \right).$$

**Theorem**

Suppose that $S$ has a universal modality. A $\Pi_2$-rule $\rho$ is admissible in $S$ iff for each simple $S$-algebra $B$ there is a simple $S$-algebra $C$ such that $B$ is a subalgebra of $C$ and $C \models \Pi(\rho)$. 
To a $\Pi_2$-rule we associate the first-order formula

$$\Pi(\rho) := \forall x, z \left( G(x) \nleq z \Rightarrow \exists y : F(x, y) \nleq z \right).$$

**Theorem**

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In the presence of a universal modality, an $S$-algebra is simple iff

$$[\forall]x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$
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**Theorem**

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In the presence of a universal modality, an $S$-algebra is simple iff

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Moreover, $S$-algebras form a discriminator variety. Therefore, the variety of $S$-algebras is generated by the simple $S$-algebras.
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Let $T$ be a universal theory in a finite language. If $T$ is **locally finite** and has the **amalgamation property**, then it admits a model completion.
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Let $T$ be a universal theory in a finite language. If $T$ is locally finite and has the amalgamation property, then it admits a model completion.

**Theorem**

Suppose that $S$ has a universal modality and let $T_S$ be the first-order theory of the simple $S$-algebras. If $T_S$ has a model completion $T_S^*$, then a $\Pi_2$-rule $\rho$ is admissible in $S$ iff $T_S^* \models \Pi(\rho)$ where

$$\Pi(\rho) := \forall x, z \left( G(x) \nless z \Rightarrow \exists y : F(x, y) \nless z \right).$$
The symmetric strict implication calculus
and contact algebras
Definition (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))

The symmetric strict implication calculus $S^2IC$ is given by the axioms

(A0) $[\forall] \phi \leftrightarrow (\top \leadsto \phi)$,
(A1) $(\bot \leadsto \phi) \land (\phi \leadsto \top)$,
(A2) $[(\phi \lor \psi) \leadsto \chi] \leftrightarrow [(\phi \leadsto \chi) \land (\psi \leadsto \chi)]$,
(A3) $[\phi \leadsto (\psi \land \chi)] \leftrightarrow [(\phi \leadsto \psi) \land (\phi \leadsto \chi)]$,
(A4) $(\phi \leadsto \psi) \rightarrow (\phi \rightarrow \psi)$,
(A5) $(\phi \leadsto \psi) \leftrightarrow (\neg \psi \leadsto \neg \phi)$,
(A8) $[\forall] \phi \rightarrow [\forall][\forall] \phi$,
(A9) $\neg [\forall] \phi \rightarrow [\forall] \neg [\forall] \phi$,
(A10) $(\phi \leadsto \psi) \leftrightarrow [\forall](\phi \leadsto \psi)$,
(A11) $[\forall] \phi \rightarrow (\neg [\forall] \phi \leadsto \bot)$,

and modus ponens (for $\rightarrow$) and necessitation (for $[\forall]$).
An open subset $A$ of a topological space is called regular open if $A = \text{int}(\text{cl}(A))$. 

Let $v$ be a valuation into a topological space $X$ that maps each propositional variable to a regular open of $X$. We can extend each valuation on all formulas as follows:

$$v(\bot) = \emptyset$$
$$v(\top) = X$$
$$v(\phi \land \psi) = v(\phi) \cap v(\psi)$$
$$v(\phi \lor \psi) = \text{int}(\text{cl}(v(\phi) \cup v(\psi)))$$
$$v(\neg \phi) = \text{int}(X \setminus v(\phi))$$
$$v(\phi \Rightarrow \psi) = \begin{cases} X & \text{if } \text{cl}(v(\phi)) \subseteq v(\psi), \\ \emptyset & \text{otherwise.} \end{cases}$$

Theorem (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))

$\vdash S2\text{IC} \phi$ iff $v(\phi) = X$ for every compact Hausdorff space $X$ and $v$. 

An open subset $A$ of a topological space is called regular open if $A = \text{int}(\text{cl}(A))$.

Let $\nu$ be a valuation into a topological space $X$ that maps each propositional variable to a regular open of $X$. We can extend each valuation on all formulas as follows

\[
\begin{align*}
\nu(\bot) & = \emptyset \\
\nu(\top) & = X \\
\nu(\phi \land \psi) & = \nu(\phi) \cap \nu(\psi) \\
\nu(\phi \lor \psi) & = \text{int}(\text{cl}(\nu(\phi) \cup \nu(\psi))) \\
\nu(\neg \phi) & = \text{int}(X \setminus \nu(\phi)) \\
\nu(\phi \Rightarrow \psi) & = \begin{cases} X & \text{if } \text{cl}(\nu(\phi)) \subseteq \nu(\psi), \\ \emptyset & \text{otherwise.} \end{cases}
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An open subset \( A \) of a topological space is called regular open if \( A = \text{int} (\text{cl}(A)) \).

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**Theorem (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))**

\[ \vdash_{S^2IC} \phi \iff \nu(\phi) = X \text{ for every compact Hausdorff space } X \text{ and } \nu. \]
The algebras associated with \( S^2 \text{IC} \) are called \textit{strict implication algebras}.

When a strict implication algebra is simple, \( \leadsto \) becomes a characteristic function of a binary relation. They correspond exactly to contact algebras.
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When a strict implication algebra is simple, $\sim\supset$ becomes a characteristic function of a binary relation. They correspond exactly to contact algebras.

**Definition**

A **contact algebra** is a boolean algebra equipped with a binary relation $\prec$ satisfying the axioms:

(S1) $0 \prec 0$ and $1 \prec 1$;
(S2) $a \prec b, c$ implies $a \prec b \land c$;
(S3) $a, b \prec c$ implies $a \lor b \prec c$;
(S4) $a \leq b \prec c \leq d$ implies $a \prec d$;
(S5) $a \prec b$ implies $a \leq b$;
(S6) $a \prec b$ implies $\neg b \prec \neg a$. 
Theorem

The model completion $\text{Con}^*$ of the theory of contact algebras is finitely axiomatizable.
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An axiomatization is given by the following three sentences.

\[
\forall a, b_1, b_2 \ (a \neq 0 \land (b_1 \lor b_2) \land a = 0 \land a \prec a \lor b_1 \lor b_2 \Rightarrow \\
\exists a_1, a_2 \ (a_1 \lor a_2 = a \land a_1 \land a_2 = 0 \land a_1 \neq 0 \land a_2 \neq 0 \land a_1 \prec a_1 \lor b_1 \land a_2 \prec a_2 \lor b_2))
\]

\[
\forall a, b \ (a \land b = 0 \land a \not\prec \neg b \Rightarrow \exists a_1, a_2 \ (a_1 \lor a_2 = a \land a_1 \land a_2 = 0 \land a_1 \not\prec \neg b \land a_2 \not\prec \neg b \land a_1 \prec \neg a_2))
\]

\[
\forall a \ (a \neq 0 \Rightarrow \exists a_1, a_2 \ (a_1 \lor a_2 = a \land a_1 \land a_2 = 0 \land a_1 \prec a \land a_1 \not\prec a_1))
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\exists a_1, a_2 \ (a_1 \lor a_2 = a & a_1 \land a_2 = 0 & a_1 \neq 0 & a_2 \neq 0 & a_1 \prec a_1 \lor b_1 \\
& a_2 \prec a_2 \lor b_2))
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\forall a, b \ (a \land b = 0 & a \not\prec \neg b \Rightarrow \exists a_1, a_2 \ (a_1 \lor a_2 = a & a_1 \land a_2 = 0 \\
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\[ \forall a, b_1, b_2 \ (a \neq 0 \& (b_1 \lor b_2) \land a = 0 \& a \prec a \lor b_1 \lor b_2 \Rightarrow \exists a_1, a_2 \ (a_1 \lor a_2 = a \& a_1 \land a_2 = 0 \& a_1 \neq 0 \& a_2 \neq 0 \& a_1 \prec a_1 \lor b_1 \& a_2 \prec a_2 \lor b_2)) \]

\[ \forall a, b \ (a \land b = 0 \& a \not\prec \neg b \Rightarrow \exists a_1, a_2 \ (a_1 \lor a_2 = a \& a_1 \land a_2 = 0 \& a_1 \not\prec \neg b \& a_2 \not\prec \neg b \& a_1 \prec \neg a_2)) \]

\[ \forall a \ (a \neq 0 \Rightarrow \exists a_1, a_2 \ (a_1 \lor a_2 = a \& a_1 \land a_2 = 0 \& a_1 \prec a \& a_1 \not\prec a_1)) \]
The following $\Pi_2$-rule

\[
\frac{(p \rightsquigarrow p) \land (\phi \rightsquigarrow p) \land (p \rightsquigarrow \psi) \to \chi}{(\phi \rightsquigarrow \psi) \to \chi}
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corresponds to the zero-dimensionality of the space.
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Using the axiomatization of $\text{Con}^*$ it is easy to show that it is admissible in $S^2IC$.  

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corresponds to the zero-dimensionality of the space.

Using the axiomatization of $\text{Con}^*$ it is easy to show that it is admissible in $S^2\text{IC}$.

Therefore, $S^2\text{IC}$ is complete wrt Stone spaces.

**Theorem (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))**

\[ \vdash_{S^2\text{IC}} \phi \iff v(\phi) = X \text{ for every Stone space } X \text{ and } v. \]
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