Modal $\mu$-calculus and alternating parity tree automata: a direct translation

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Introduction

About the modal $\mu$-calculus:

- useful for model checking
- contains temporal logics LTL, CTL...
- equivalent to monadic second order logic over trees
- equivalent automata models allow to reason more easily on its semantics using the parity condition (the one from parity games)
Bibliography

- [EJ91] An article from E. Allen Emerson and Charanjit S. Jutla which gives the translation from $\mu$-calculus to tree automata only for the Streett condition which is equivalent to the parity one. We wanted parity condition and there is no converse translation.

- [JW95] An article from Janin and Walukiewicz which gives a complete translation but only for disjunctive formulas. They prove it is always possible to put a formula in disjunctive form, so there is an equivalence but the translation is not direct and the proof is quite short, we wanted more details.

- [Wil01] An article from Thomas Wilke which translates $\mu$-calculus to automata, but not the converse. The automata formalism is quite different than ours.
Objectives of our direct translation

We do not know any direct ”bidirectional” translation between $\mu$-calculus formulas and the alternating parity tree automata we will present later on.

Our aim is to use this direct translation to rewrite the Kobayashi-Ong type system [KO09] in a way that will reveal a connection with cyclic proofs.
Tree and signature

We consider $\Sigma$-labelled ranked tree defined with a signature. Here is an example defined over the signature $\Sigma = \{a : 2, b : 1, c : 0\}$:

```
   a
  /   \
 c     a
 / \\  /  \
 b   a b a
 / \ / \ / \
 c b c b c
```

...
Modal $\mu$-calculus : syntax

Consider a signature $\Sigma = \{ a : ar(a), b : ar(b), c : ar(c)\ldots \}$ and a set of variable $Var = \{ X, Y, Z\ldots \}$.

The grammar is defined inductively:

$$\varphi ::= X \mid a \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid \square \varphi \mid \lozenge_i \varphi \mid \lozenge \varphi \mid \mu X.\varphi \mid \nu X.\varphi$$

Example of formulas:

- $\mu X.\lozenge X$
- $\nu X. (\square X \land \mu Y.(a \lor \lozenge Y \lor \nu Z.(b \land \square Z)))$
Modal $\mu$-calculus: semantics

Consider a valuation function $V : \text{Var} \rightarrow \mathcal{P}(N)$, then:

- $|a|_V = \{ n \in N \mid n \text{ is labelled by } a \}$
- $|X|_V = V(X)$
- $|\neg \varphi|_V = N \setminus |\varphi|_V$
- $|\varphi \lor \psi|_V = |\varphi|_V \cup |\psi|_V$
- $|\varphi \land \psi|_V = |\varphi|_V \cap |\psi|_V$
- $|\diamondsuit_i \varphi|_V = \{ n \in N \mid \text{ar}(n) \geq i \text{ and } \text{succ}_i(n) \in |\varphi|_V \}$
- $|\mu X. \varphi(X)|_V = \bigcap\{ M \subset N \mid |\varphi|_V[X \leftarrow M] \subset M \}$
- $|\nu X. \varphi(X)|_V = \bigcup\{ M \subset N \mid M \subset |\varphi|_V[X \leftarrow M] \}$
Example of semantics

$$\varphi_1 = \| \nu X.(\Diamond X \lor c) \|
\varphi_2 = \| \mu X.(\Box X \lor c) \|$$
Alternating parity tree automata (APTA) over an example

We consider an APTA $A$ with states $q_0$ and $q_1$, initial states $q_0$:

$\delta(a, q_0) = (1, q_0) \land (1, q_1) \land (2, q_0)$:
Alternating parity tree automata (APTA) over an example

If we add the following transitions: \( \delta(b, q_0) = (1, q_0) \lor (1, q_1), \delta(b, q_1) = (1, q_0) \lor (1, q_1), \delta(c, q_0) = \top, \delta(c, q_1) = \top \) \} One possible transcription is:

\[
\begin{array}{c}
\text{a} \\
\text{c} \quad \text{a} \\
\text{b} \quad \text{a} \\
\text{c} \quad \text{b} \quad \text{a} \\
\vdots \\
\text{c} \\
\end{array}
\]

\[
\begin{array}{c}
a_{q_0} \\
c_{q_0} \quad c_{q_1} \quad a_{q_0} \\
b_{q_0} \quad b_{q_1} \quad a_{q_0} \\
c_{q_0} \quad c_{q_0} \quad b_{q_0} \quad b_{q_1} \quad a_{q_0} \\
\vdots \quad \vdots \\
c_{q_0} \quad c_{q_0} \quad b_{q_1} \quad b_{q_1} \\
\end{array}
\]
Alternating parity tree automata (APTA) over an example

We use the coloring $\Omega(q_0) = 0$ and $\Omega(q_1) = 1$.

The infinite branch is accepted.
Translation from the modal $\mu$-calculus to APTA

- We translate it by induction on the formula syntax.
- Every state of the resulting automaton correspond to a subformula.
- For the coloration of the state, we use the alternation depth.

**Example**

The following formula has alternation depth 2:

$$\nu X. (\square X \land \mu Y.(a \lor \diamond Y \lor \nu Z.(b \land \square Z)))$$

The following formula has alternation depth 1:

$$\nu X. (\square X \lor \mu Y.(a \lor \diamond Y) \lor \nu Z.(b \land \diamond Z))$$
Translation from APTA to $\mu$-calculus without parity condition

Principle: We put the transition functions in disjunctive form as for example $\delta(q_k, a_k) = \bigvee_{i \in I_k} \bigwedge_{j \in J_k} (d_{i,j}, q_{i,j})$ Each state correspond to a formula, the initial state is the resulting formula.

$$X_k = \bigvee_{a \in \Sigma} \bigwedge_{i \in I_a} \bigwedge_{j \in J_a} \diamond_{d_{i,j}} X_{i,j}$$

Example

We consider the signature $\Sigma = \{a : 2\}$ and the automaton $A = \langle \Sigma, Q = \{q_1, q_2\}, \delta = \{(q_1, a) : (1, q_2) \land (2, q_2), (q_2, a) : (1, q_1) \land (2, q_1)\}, q_1, \Omega = \{q_1 : 1, q_2 : 4\}\rangle$. We have:

$$X_1 = a \land \diamond_1 X_2 \land \diamond_2 X_2$$

$$X_2 = a \land \diamond_1 X_1 \land \diamond_2 X_1$$
Translation from APTA to $\mu$-calculus: adding the parity condition

We add colored equalities with the same color that the state considered:

$$X_1 =_1 (a \land \Diamond_1 X_2 \land \Diamond_2 X_2)$$
$$X_2 =_4 (a \land \Diamond_1 X_1 \land \Diamond_2 X_1)$$

We say this system of equations is not \textbf{parity coherent}: there is a recursive call to the color 4 under the scope of a recursive call of color 1. This is problematic to obtain the translation to modal $\mu$-calculus formulas.
Obtaining parity coherence

We can use the rules to obtain a parity coherent term.

Set $X_1 = \rho_1 X_1 . (a \land \diamond_1 X_2 \land \diamond_2 X_2) = \mathcal{R}(X_1)$.

Then $X_1$ rewrites to:

$$a \land \diamond_1 (X_2[X_1 \leftarrow \mathcal{R}(X_1)]) \land \diamond_2 (X_2[X_1 \leftarrow \mathcal{R}(X_1)])$$

emitting the color 1 (negligible from the point of view of the parity condition), and then this rewrites to:

$$a \land \diamond_1 (\rho_4 X_2 . (a \land \diamond_1 \mathcal{R}(X_1) \land \diamond_2 \mathcal{R}(X_1))) \land \diamond_2 (\rho_4 X_2 . (a \land \diamond_1 \mathcal{R}(X_1) \land \diamond_2 \mathcal{R}(X_1)))$$

which is parity coherent.
Conclusion

- We prove the equivalence between $\mu$-calculus and APTA, and we give a direct translation from one structure to another.

- We use as an intermediate step the notion of parity coherence, that does not appear with other equivalent acceptance condition such as the Streett condition.
References

