Impossible states at work : Logical omniscience and rational choice

Mikaël Cozic*

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(Paris IV-Sorbonne, Department of Philosophy, and IHPST (CNRS-Paris I))

Introduction

It is well-known that usual doxastic models (epistemic logic, probability) suffer from strong cognitive idealizations. By extension, models of decision making that elaborate on these doxastic models (e.g., models of choice under set-theoretic uncertainty ([LR85], chap.13) and expected utility) inherit these idealizations. To improve doxastic models is therefore an important aspect of bounded rationality.

The most uneliminable of these idealizations is probably logical omniscience : in doxastic models, beliefs are closed under the consequence relation of classical logic. Any agent so modelled is supposed to believe all the consequences of his or her beliefs.

There has been extensive works to solve this problem in the framework of epistemic logic ([FHMV95]), which corresponds approximately to set-theoretic models in the rational choice literature[1]. Surprisingly, there have been few attempts to investigate the extension of the putative solutions to the probabilistic representation of beliefs (*probabilistic case*) and to models of decision making (*decision-theoretic case*)[2].

The aim of this paper is to fill this gap.

The remainder of the paper proceeds as follows. In section 1, the problem of logical omniscience and its most popular solutions are briefly recalled. Then, it shall be argued that, among these solutions, non-standard structures are the best basis for an extension to probabilistic and decision-theoretic cases. Section 2 is devoted to the probabilistic case, and section 3 to the decision-theoretic case. We conclude in section 4.

*Thanks to D.Andler, D.Bonnay and B.Walliser for helpful comments.

[1]More precisely, it is akin to the information partitions in game theory, see [Aum99].

[2]One of the most recent exception is [Lip99]

1 Logical omniscience in epistemic logic

1.1 Epistemic logic

Problems and propositions related to logical omniscience are best expressed in a logical framework, usually called "epistemic logic". Here is a brief review of the classical model : Kripke structures.

First, we have to define the *language* of propositional epistemic logic. The only difference with the language of propositional logic is that this language contains a doxastic operator B : $B\phi$ is intended to mean "the agent believes that ϕ ".

Definition 1
The set of ***formulas*** *of an epistemic propositional language* $\mathcal{L}B(At)$, $Form(\mathcal{L}B(At))$, *is the subset of* $\mathcal{L}B(At)$ *such that*

(i) if $p \in At$, $p \in Form(\mathcal{L}B(At))$,

(ii) if $\phi \in Form(\mathcal{L}B(At))$, *then* $\neg\phi \in Form(\mathcal{L}B(At))$,

(iii) if $\phi, \psi \in Form(\mathcal{L}B(At))$, *then* $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi) \in Form(\mathcal{L}B(At))$, *and*

(iv) if $\varphi \in Form(\mathcal{L}B(At))$, *then* $B\varphi \in Form(\mathcal{L}B(At))$.

(v) only strings of symbols generated by (i)-(iv) in a finite number of steps are in $Form(\mathcal{L}B(At))$.

$Form(\mathcal{L}(At))$ is defined in the same way, except that we drop condition (iv).

Second, we have to give an interpretation (or semantics) to the language, *ie* we have to give the formulas' truth-conditions. A language is interpreted by means of a Kripke structure.

Definition 2
Let $\mathcal{L}B(At)$ *an epistemic propositional language ; a* ***Kripke structure*** *for* $\mathcal{L}B(At)$ *is a 3-tuple* $\mathcal{M} = (S, \pi, R)$ *where*

(i) S *is a state space,*

(ii) $\pi : At \times S \rightarrow \{0, 1\}$ *is a valuation*

(iii) $R \subseteq S \times S$ *is an accessibility relation*

Intuitively, the accessibility relation associates to every state the states that the agent considers possible given his or her beliefs. π associates to every atomic formula, in every state, a truth value ; it is extended in a canonical way to every formula by the satisfaction relation.

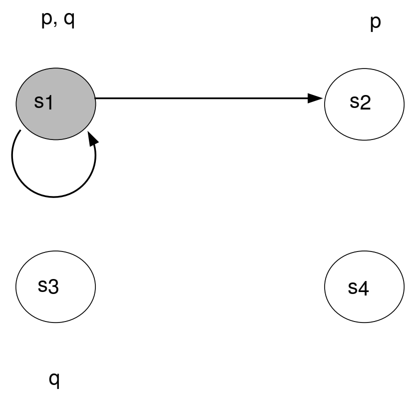

Figure 1: Kripke structures

Definition 3
*$\bar{\pi}$, called the **satisfaction relation**, extends π to every formula of the langage according to the following conditions :*

(i) $\bar{\pi}(s,p) = \pi(s,p)$ *if* $p \in At$

(ii) $\bar{\pi}(s, \phi \wedge \psi) = 1$ *iff* $\bar{\pi}(s,\phi) = 1$ *and* $\bar{\pi}(s,\psi) = 1$

(iii) $\bar{\pi}(s, \phi \vee \psi) = 1$ *iff* $\bar{\pi}(s,\phi) = 1$ *or* $\bar{\pi}(s,\psi) = 1$

(iv) $\bar{\pi}(s, \neg\phi) = 1$ *iff* $\bar{\pi}(s,\phi) = 0$

(v) $\bar{\pi}(s, B\phi) = 1$ *iff* $\forall s'$ *s.t.* sRs', $\bar{\pi}(s,\phi) = 1$

The specific doxastic condition contains what might be called the **possible-state analysis of belief**. It means that an agent believes that ϕ if, in all the states that could be according to him or her the actual state, ϕ is true : *believe is to exclude that it could be false.* Conversely, an agent doesn't believe ϕ if, in some of the states that could be the actual state, ϕ is false : *not to believe is to consider that it could be false.* This principle will be significant in the discussions below.

Example 1
$S = \{s_1, s_2, s_3, s_4\}$; p ("it's sunny") is true in s_1 and s_2, q ("it's windy") in s_1 and s_4. Suppose that s_1 is the actual state and that in this state the agent believes that p is true but does not know if q is true. Figure 1 represents this situation, omitting the accessibility relation in the non-actual states.

Definition 4
*Let $\mathcal{M}$ be a Kripke structure ; in $\mathcal{M}$, the set of states where ϕ is true, or the **proposition** expressed by ϕ, or the **informational content** of ϕ, is noted* $[[\varphi]]_{\mathcal{M}} = \{s : \bar{\pi}(\varphi, s) = 1\}$.

To formulate logical omniscience, we need lastly to define the following semantical relations between formulas.

Definition 5
*ϕ **$\mathcal{M}$-implies** ψ if $[[\phi]]_{\mathcal{M}} \subseteq [[\psi]]_{\mathcal{M}}$. ϕ and ψ are **$\mathcal{M}$-equivalent** if $[[\phi]]_{\mathcal{M}} = [[\psi]]_{\mathcal{M}}$*

There are several forms of logical omniscience (see [FHMV95]) ; the next proposition shows that two of them, deductive monotony and intensionality, hold in Kripke structures :

Proposition 1

Deductive monotony : if ϕ $\mathcal{M}$-implies ψ, then $B\phi$ $\mathcal{M}$-implies $B\psi$

Intensionality : if ϕ and ψ are $\mathcal{M}$-equivalent, then $B\phi$ and $B\psi$ are $\mathcal{M}$-equivalent

1.2 Three putative solutions to logical omniscience

1.2.1 Neighborhood structures

Definition 6
*A **neighborhood structure** is a 3-tuple $\mathcal{M} = (S, \pi, V)$ where*

(i) S is a state space,

(ii) $\pi : At \times S \rightarrow \{0, 1\}$ is a valuation,

*(iii) $V : S \rightarrow \wp(\wp(S))$, called the agent's **neighborhood system**, associates to every state a set of propositions.*

The conditions on the satisfaction relation are the same, except for the doxastic operator :

$\bar{\pi}(B\phi, s) = 1$ iff $[[\phi]]_{\mathcal{M}} \in V(s)$

The innovation of neighborhood structures consists in making explicit, in each state, all the propositions that the agent believes. It's easy to check that deductive monotony is invalidated by neighborhood structures, as shows the following example.

Example 2
Let's consider the first example and replace the accessibility relation by a neighborhood system ; $V(s_1)$ contains $\{s_1, s_2\}$ but not $\{s_1, s_2, s_3\}$. Then, in s_1, Bp is true but not $B(p \vee q)$. This is represented in Figure 2.

Therefore, deductive monotony is easily controlled in neighborhood structures: by the closure of neighborhood sets under supersets. In contrast, the main limitation of neighborhood structures is that intensionality cannot be weakened.

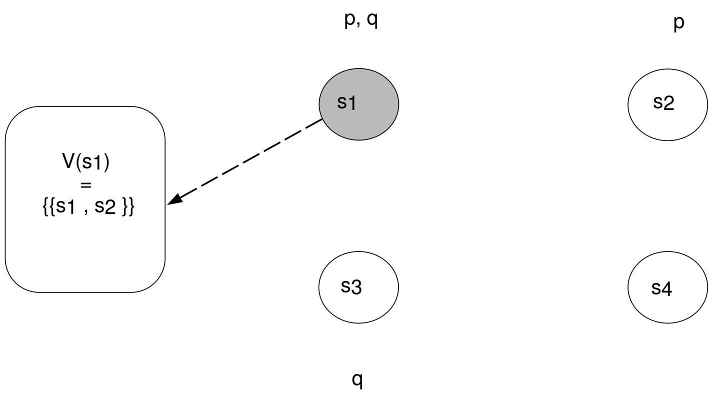

Figure 2: Neighborhood structures

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1.2.2 Awareness structures

Definition 7
*An **awareness structure** is a 4-tuple* (S, π, R, A) *where*

(i) S *is a state space,*

(ii) $\pi : At \times S \rightarrow \{0, 1\}$ *is a valuation,*

(iii) $R \subseteq S \times S$ *is an accessibility relation,*

(iv) $A : S \rightarrow Form(\mathcal{L}B(At))$ *is a function which maps every state in a set of formulas ("awareness set").*

The new condition on the satisfaction relation is the following :

$\bar{\pi}(B\phi, s) = 1$ iff $\forall s'$ s.t. sRs', $s' \in [[\phi]]_{\mathcal{M}}$ and $\phi \in A(s)$

According to this new satisfaction relation, an agent believes a formula not only if the formula is true in all the states compatible with his or her beliefs, but if the formula is in the agent's awareness set too. This awareness condition permits to weaken any form of logical omniscience.

Example 3
Let's consider our example and stipulate that $A(s_1) = \{p\}$*. Then it is still the case that* Bp *is true in* s_1*, but not* $B(p \vee q)$*. This is represented in Figure 3.*

1.2.3 Non-standard structures

Definition 8
*A **non-standard structure** is a 4-tuple* $\mathcal{M} = (S, S', \bar{\pi}, R)$ *where*

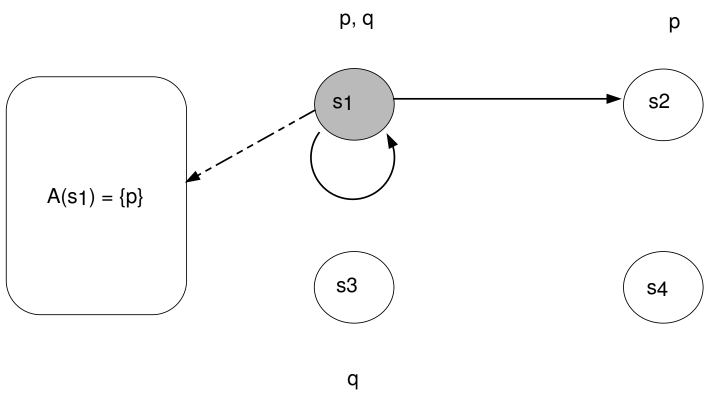

Figure 3: Awareness structures

(i) S is a space of standard states,

(ii) S' is a space of non-standard states,

(iii) $R \subseteq S \cup S' \times S \cup S'$ is an accessibility relation,

(iv) $\pi : Form(\mathcal{L}B(At)) \times S \rightarrow \{0, 1\}$ is a satisfaction relation ***standard on*** *S*

In non-standard structures, there are no constraints on the satisfaction relation in non-standard states. For instance, in a non-standard state, both ϕ and $\neg\phi$ can be false. For every formula ϕ, one might therefore distinguish its *objective informational content* $[[\phi]]_{\mathcal{M}} = \{s \in S : \pi(\phi, s) = 1\}$ from its *subjective informational content* $[[\phi]]^*_{\mathcal{M}} = \{s \in S^* = S \cup S' : \pi(\phi, s) = 1\}$

In spite of appearances, this generalization of Kripke structures is arguably natural for logical omniscience as soon as one accepts the possible-state analysis of beliefs. According to this analysis,

- to believe that ϕ is to exclude that ϕ could be false, and
- not to believe that ψ is not to exclude that ψ could be false.

In consequence, according to this analysis, to believe that ϕ but not to believe one of its consequence ψ is to consider as possible at least one state where ϕ is true but ψ false. By definition, a state of this kind is logically non-standard.

Example 4
Let's consider our example but add a non stantard state in $S' = s_5$; $\bar{\pi}(p, s_5) = 1$, $\bar{\pi}(p \vee q, s_5) = 0$. Then in s_1, Bp is true but not $B(p \vee q)$. This is represented in Figure 4.

2 The probabilistic case

In this section, we study the probabilistic extension of doxastic models without logical omniscience.

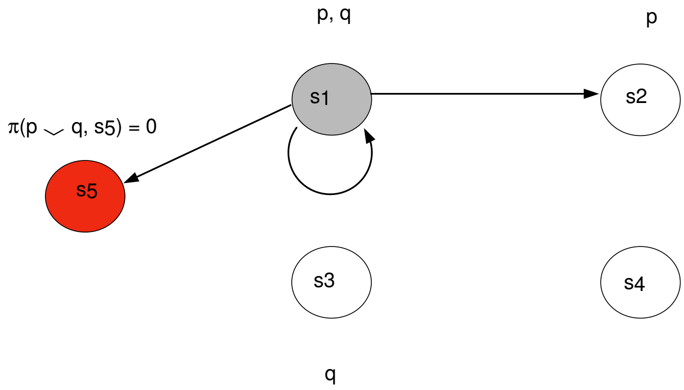

Figure 4: Non-standard structures

2.1 Probabilistic counterpart of logical omniscience

First, we have to define the probabilistic counterparts of logical omniscience ; that the definitions below are indeed *counterparts* of logical omniscience can be checked in the limit case of certainty, *ie* when the agent believes relevant formulas to degree 1. In the usual (non-logical) framework, if P is a probability distribution on S,[3] then the following properties are the counterparts of logical omniscience :

- if $E \subseteq E'$, then $P(E) \leq P(E')$,

- if $E = E'$, then $P(E) = P(E')$.

But to be closer to the preceding section, it is better to work with an elementary[4] logical version of the usual probabilistic model :

Definition 9
*Let $\mathcal{L}(At)$ a propositional language ; a **probabilistic structure**[5] for $\mathcal{L}(At)$ is a 3-tuple $\mathcal{M} = (S, \pi, P)$ where*

(i) S is a state space,

(iii) π is a valuation,

(iv) P is a probability distribution on S.

An agent believes to degree r a formula $\phi \in Form(\mathcal{L}(At))$, symbolized by $CP(\phi) = r$, if $P([[\phi]]_{\mathcal{M}}) = r$

Proposition 2
The following holds in probabilistic structures :

[3] We suppose that S is finite and that P is defined on $\wp(S)$.
[4] "Elementary" because there is no doxastic operator in the object-language.
[5] see [FH91].

- deductive monotony : if ϕ $\mathcal{M}$-implies ψ, then if $CP(\phi) \geq CP(\psi)$.

- intensionality : if ϕ and ψ are $\mathcal{M}$-equivalent, then $CP(\phi) = CP(\psi)$.

2.2 The case against neighborhood and awareness structures

Now, the issue we would like to address is this : which of the three putative solutions to logical omniscience should we try to extend to the probabilistic case ? Consider first the neighborhood structures. One could imagine an extension of the following kind : to each possible degree of belief r is assigned the set of propositions that the agent believes to this degree. But there are at least three strong reasons not to follow this way : (i) as in the case of epistemic logic, neighborhood structures are not strong enough : they cannot weaken intensionnality, (ii) as we shall see in the next section, an extension to the decision-theoretic case is hard to conceive, and (iii) the doxastic state is modeled by a quite complex function. Matters are worse for the awareness structures : it is even difficult to conceive any extension of the set-theoretic case.

In contrast, those troubles don't arise for the non-standard structures.

2.3 Probabilistic non-standard structures

Definition 10
*Let $\mathcal{L}(At)$ a propositional language ; a **non-standard probabilistic structure** for $\mathcal{L}(At)$ is a 4-tuple $\mathcal{M} = (S, S', \pi, P)$ where*

(i) S is a standard state space,

(ii) S' is a non-standard space,

(iii) $\pi : Form(L(At)) \times S \cup S' \rightarrow \{0, 1\}$ is a satisfaction relation which is standard on S,

(iv) P is a probability distribution on $S^* = S \cup S'$.

As in the set-theoretic case, one can distinguish the objective informational content of a formula, *ie* the standard states where this formula is true, and the subjective informational content of a formula, *ie* the non-standard states where this formula is true.

To obtain the expected benefit, the non-standard probabilistic structures should characterize the agent's doxastic state on the basis of *subjective* informational content : an agent believes a formula ϕ to degree r, $CP(\phi) = r$, if $P([[\phi]]^*_{\mathcal{M}}) = r$.

It is easy to check that, in this case, logical omniscience can be utterly controlled.

Example 5
Let's take the same space state as in the preceding examples. Suppose that the agent has the following partial beliefs : $CP(p) > CP(p \vee q)$. This can be

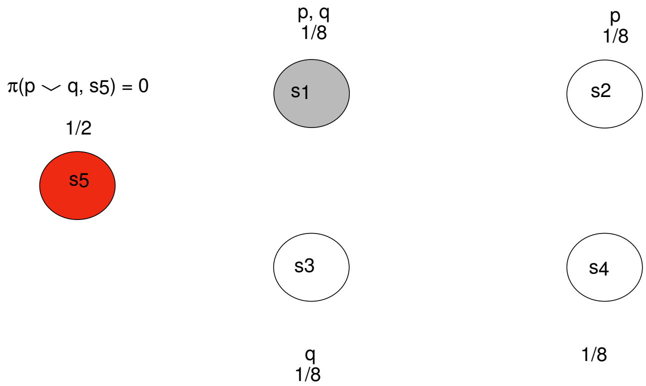

Figure 5: Probabilistic non-standard structures

modelled in the following way : $S' = \{s_5\}$*,* $s_5 \in [[p]]^*_{\mathcal{M}}$ *but* $s_5 \notin [[p \vee q]]^*_{\mathcal{M}}$*,* $P(s_1) = P(s_2) = P(s_3) = P(s_4)1/8$ *and* $P(s_5) = 1/2$*. This is represented in Figure 5.*

2.4 Special topics : deductive information and additivity

This extension of non-standard structures is admittedly straightforward and simple ; it gives immediately the means to weaken logical idealizations. Furthermore, it opens perspectives specific to the probabilistic case ; two of them will be briefly mentioned.

Deductive information and learning First, one can model the fact that an agent acquire not only empirical information but *deductive information* ; in non-standard structures, this corresponds to the fact that the agent eliminates non-standard states.

Let's come back to our generic situation. Suppose that our agent learns that ϕ implies ψ. This means that he or she learns that the states where ϕ is true but ψ false are impossible. This is equivalent to say that he or she learns the event

$$I = S^* - ([[\phi]]^*_{\mathcal{M}} - [[\psi]]^*_{\mathcal{M}})$$

To be satisfying, such a notion of deductive information must respect a requirement of compatibility between conditionalization and logical monotony : if the agent learns that ϕ implies ψ and conditionalize upon this fact, his or her new probability distribution should conform to logical monotony with respect to ϕ and ψ. One can check that it is the case.

Proposition 3
If I *is learned following the Bayes ryle, then deductive monotony is regained,* ie $CP_I(\phi) \leq CP_I(\psi)$.

Example 6
This can be checked in the preceding example : $I = S = \{s_1, s_2, s_3, s_4\}$*. By conditionalization,* $CP_I(p) = 1/2$ *whereas* $CP_I(p \vee q) = 3/4$*.*

Additivity A second topic is additivity. From a logical point of view, one can define additivity as follows :

Definition 11
$\mathcal{M}$ *is (logically)* ***additive*** *if, when* ϕ *and* ψ *are logically incompatible,* $CP(\phi) + CP(\psi) = CP(\phi \vee \psi)$*.*

Additivity is of course the core of the probabilistic representation of beliefs, and alternative representations of beliefs depart often from probability on this point. For example, in the Dempster-Shafer theory ([Sha76]), the so-called belief function is superadditive (in our notation, $CP(\phi \vee \psi) \geq CP(\phi) + CP(\psi)$) whereas its dual, the plausibility function, is subadditive ($CP(\phi \vee \psi) \leq CP(\phi) + CP(\psi)$).

A noteworthy aspect of probabilistic non-standard structures is that the freedom of the connectives' behavior in non-standard states permits us to have a very flexible framework with respect to additivity : simple conditions on the connectives imply general properties concerning additivity.

Definition 12
Let $\mathcal{M} = (S, S', \pi, P)$ *a probabilistic non-standard structure ;* $\mathcal{M}$ *is* $\vee$***-standard*** *if for every formulas* ϕ, ψ, $[[\phi \vee \psi]]^*_{\mathcal{M}} = [[\phi]]^*_{\mathcal{M}} \cup [[\psi]]^*_{\mathcal{M}}$*.*

This means that the disjonction behaves in the usual way in non-standard states ; a trivial consequence of this is that the structure $\mathcal{M}$ is (logically) subadditive.

Proposition 4
If $\mathcal{M}$ *is* $\vee$*-standard, then it is (logically) subaddditive.*

To be more general, one can consider the (logical) inclusion-exclusion rule :

$$CP(\phi \vee \psi) = CP(\phi) + CP(\psi) - CP(\phi \wedge \psi)$$

One can define (logical) **submodularity** (resp. supermodularity or convexity) as : $CP(\phi \vee \psi) \leq CP(\phi) + CP(\psi) - CP(\phi \wedge \psi)$ (resp. $CP(\phi \vee \psi) \geq CP(\phi) + CP(\psi) - CP(\phi \wedge \psi)$).

It's clear that to control submodularity, we have to control the conjunction's behavior.

Definition 13
Let $\mathcal{M} = (S, S', \pi, P)$ *a probabilistic non-standard structure ;*

- $\mathcal{M}$ *is* ***negatively*** $\wedge$***-standard*** *if for every formulas* ϕ, ψ*, when* $\pi(\phi, s) = \pi(\psi, s) = 0$ *or* $\pi(\phi, s) \neq \pi(\psi, s)$*, then* $\pi(\phi \wedge \psi) = 0$*.*

- $\mathcal{M}$ *is* ***positively*** $\wedge$***-standard*** *if for every formulas* ϕ, ψ*, when* $\pi(\phi, s) = \pi(\psi, s) = 1$*, then* $\pi(\phi \wedge \psi) = 1$*.*

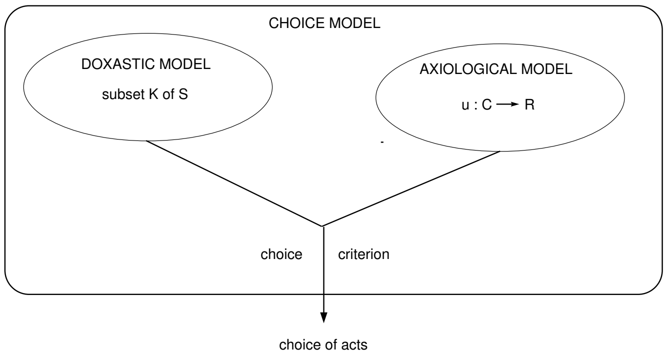

Figure 6: The structure of choice model

Proposition 5
Suppose that $\mathcal{M}$ is $\vee$-standard ;
- if $\mathcal{M}$ is negatively $\wedge$-standard, then submodularity holds.
- if $\mathcal{M}$ is positively $\wedge$-standard, then supermodularity holds.

Proof : see the Appendix.

3 The decision-theoretic case

This final section is devoted to the decision-theoretic case; indeed, to concentrate on conceptual issues, only the models of decision under set-theoretic uncertainty will de considered. It is worthy to note that [Lip99] has developed a model of choice under probabilistic uncertainty with non-standard states.

The models of choice under uncertainty we shall consider have in their usual form the following structure : a set A of opportunities, a state space S and a set $K \subseteq S$ representing the states that, according to the agent, can be the actual state, a set C of consequences, a consequence function $\mathcal{C} : A \times S \to C$ and an utility fonction $u : C \to \mathbb{R}$.

The models of decision making can be roughly conceived as the composition of two main submodels :

(i) *a doxastic model* : in this case, the set of possible states $K \subseteq S$ compatible with the agent's beliefs,

(ii) *an axiological model* : in this case, the utility function u.

The "composition" is achieved by the criterion of choice, for example the *maximin* criterion. This is represented in Figure 6.

The preceding sections have shown that the usual doxastic models are very idealized but that it is possible to improve them. To this point, important issues arise : is it equally possible for the alternatives doxastic models to have the function of submodel in a model of choice ? And if yes, how ?

Our claim is the following : if one integrates neighborhood or awareness structures, one faces a deep dilemma which has no straightforward solution. This dilemma can be avoided with non-standard structures.

3.1 The case against neighborhood and awareness structures

The case is probably clearer for the neighborhood structures : suppose that the doxastic state of an agent is modeled by such a structure. This means that a set of propositions is associated to the agent in each state. If the agent is not logically omniscient, sometimes he will have a proposition in the set but not all supersets of this proposition. Let us suppose lastly, for convenience, that there is a smallest proposition in the belief set of the agent. How to elaborate a choice model on this doxastic submodel ?

One can conceive the usual criterion as a function that takes as input the states the agent considers possible. How to apply the criterion in the case of neighborhood structures ?

The first option is conservative and tries to find a set of states to apply the criterion ; obviously, the modeler will choose the smallest set. But it's easy to see that in this case the logical weakness of the agent will have no consequences on his or her decisions.

The second option is to try to make the criterion more sensitive to the doxastic model. In this case, this means that the criterion should take as argument the entire set of propositions. But it is not at all clear to see how to extend the criterion.

Briefly stated, either one maintains the criterion but neglects the idealizations or one tries to generalize the criterion but probably loses the naturalness of the original criterion.

The dilemma is still stronger for awareness structures : either one applies the criterion to the accessibility set and clearly the logical abilities of the agent cannot change anything ; or one tries to take into account the awareness set, but then one has to build a syntactic-sensitive criterion.

The main power of non-standard doxastic structures is that they escape this dilemma : one can both keep the usual decision criteria *and* model the effects on choices of differences in logical competence.

3.2 Non-standard models of choice

In the last part of this section, we propose an example of non-standard model of choice. This example is quite specific but the motivation underlying it is to give an insight of the non-standard models' potentialities.

The target of this sample model is the situation where an agent knows in principle the consequence function, *ie* knows in principle what follows from each pair $(action, state)$, but is not able to infer from that the exact value of each argument. We suppose that he bases his or her choice on those exact values.

As an instance of not-so-simple consequence function, let's take a classic two-state example of insurance application [6]. The consequence function is

$\mathcal{C}(s_1, x) = w - \pi x$

[6] From [LM81].

$\mathcal{C}(s_2, x) = y + x$,

where x, the choice variable, is the amount of money spent in insurance, s_1 the state without disaster, w the wealth in s_1, s_2 the state with a disaster and y the subsequent wealth, and a least π the rate of exchange.

Let's start from a slight variant of the above mentioned model : instead of having separated state space S and consequence set C, the model is based on extended states $w \in W$. An **extended state** contains a state of nature and the consequences of the available acts in this state. This second component is captured by a local consequence function, $\mathcal{C} : A \to C$. An extended state is therefore a pair $(s, \mathcal{C}_w)$. Let $\mathcal{C}^T : A \times S \to C$ be the true consequence function, *ie* the consequence function known, in principle, by the agent.

Definition 14
An extended state $w = (s, \mathcal{C}_w)$ is ***standard*** *if for all $a \in A$, $\mathcal{C}_w(a) = \mathcal{C}^T(a, s)$.*

Now, we can see how to model our target situation : according to the possible-state analysis of belief, if, for example, an agent does not know the exact consequence of an act a given a state s, then we stipulate at least a non-standard extended state $w' = (s, \mathcal{C}'_w)$ where $\mathcal{C}_{w'}(a) \neq \mathcal{C}^T(a, s)$.[7] This is represented in Figure 7 : the target situation is at the top, where an agent does not know what is the precise consequence of action 2 in state s_1.

What is described above constitutes a non-standard doxastic model appropriate for the usual models of decision making. On such a basis, one can apply his or her favorite choice criterion to this non-standard model.

For example, if one takes the maximin criterion, one obtains the *non-standard model maximin* (MaxMinNS) and the model's solution is :

$$Sol_{MaxMinNS} = \arg\max_{a \in A} \min_{w \in K} u(\mathcal{C}_w(a))$$

Starting from this insight, different issues can be explored.

If one remains in the set-theoretic uncertainty framework, one might study precisely the effects on choices of logical incompetence, especially the logical incompetence's *costs*.

Another significant issue is of course *the extension to expected utility* : the model of conditional expected utility [Lip99] is very specific, and, for instance, it would be interesting, from an axiomatic point of view, to compare the classical model of Savage to a non-standard version. Savage has shown that

(i) given S and C,

(ii) if certain conditions on the preference relation $\preceq$ on $F = C^S$ hold,

[7] In other words, this means that, in this particular case, one might roughly consider that logical ignorance is modeled by using *correspondences* instead of functions to represent acts : to every pair $(act, state)$ is associated a set of consequences. The use of correspondences has recently been explored in the subjective expected utility model by [Ghi01]. Given a "true" consequence function, the equivalence between non-standard states and correspondences can be made precise in the set-theoretic case : if the choice criterion evaluates actions according to their consequences, then to every non-standard model can be associated a "correspondence model" that gives the same solution, and conversely.

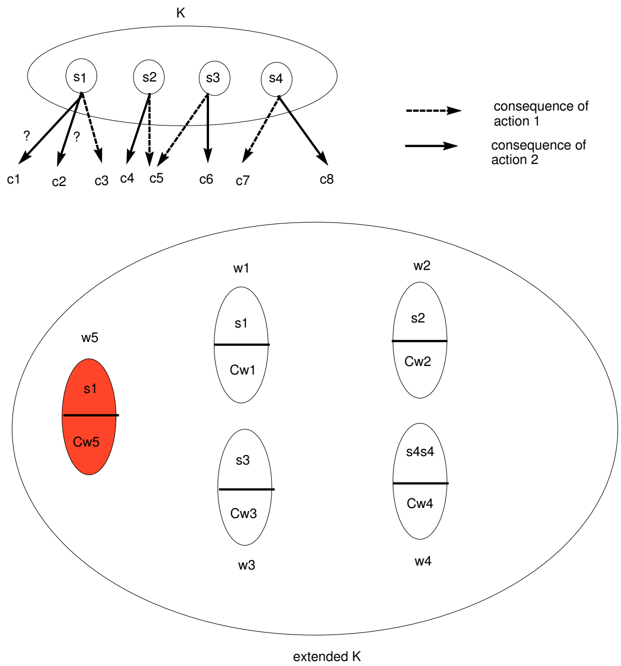

Figure 7: Non-standard extended states

(iii) then there exists a probability distribution P on S and an utility function u on C s.t. $\preceq$ is SEU-representable.

In our case, the following question could be asked : given S and C, to which conditions on $\preceq$ does there exist *a non-standard state space* $S^* \supseteq S$, a probability distribution P^* on S^* and an utility function s.t. $\preceq$ is SEU-representable ?

The question is worth being asked, because it seems, for instance, that some violations of the Sure-Thing Principle can be rationalized by non-standard states :

(STP) $(f =_E f', g =_E g', f =_{E^c} g$ and $f' =_{E^c} g')$ implies $(f \preceq g$ iff $f' \preceq g')$.

This is the object of current research.

4 Conclusion

The aim of this paper was to show that non-standard states is the most promising tool to extend the weakening of doxastic models' idealizations. A further argument is given in a companion paper where it is shown that unawareness can be modelled by non-standard states too.

Of course, these non-standard models need further explorations. But we would like to conclude by stressing two important limitations.

First, non-standards are frameworks, and not theories : they give the means to express more realistic cognitive states, but they don't contain any substantial theories concerning the real cognitive states.

Second, logical omniscience is concerned with deductive idealization ; but all deductive idealizations involved in the rational choice model is not controlled by the kind of model exposed above. To be more precise, the deductive idealization implicit in the choice criterion is not up to our proposal : whether or not the agent has an imperfect state space, he is still perfectly able to find the acts that satisfy the criterion. How to improve this point ? We leave this as an open question.

5 Appendix

The proof deals only with the case of submodularity ; the other is symetric.

If $[[\phi]]^*$ and $[[\psi]]^*$ are disjoint, then by hypothesis $[[\phi \wedge \psi]]^* = \emptyset$. Therefore $CP(\phi \vee \psi) = CP(\phi) + CP(\psi) - CP(\phi \wedge \psi)$.

It follows from the definition that if $\pi(\psi \wedge \phi, s) = 1$, then $\pi(\psi, s) = \pi(\phi, s) = 1$ (the converse does not hold). In other words,

(1) $[[\phi \wedge \psi]]^* \subseteq [[\phi]]^* \cap [[\psi]]^*$.

This implies that

(2) $P([[\phi \wedge \psi]]^*) \leq P([[\phi]]^* \cap [[\psi]]^*)$.

Since $\mathcal{M}$ is $\vee$-standard, $P([[\phi \vee \psi]]^*) = P([[\phi]]^*) + P([[\phi]]^*) - P([[\phi]]^* \cap [[\psi]]^*)$. By (2), it follows from this that

$P([[\phi \vee \psi]]^*) \leq P([[\phi]]^*) + P([[\phi]]^*) - P([[\phi \wedge \psi]]^*)$.

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