First-Order Interpolation Derived from Propositional Interpolation*

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Ever since Craig's seminal paper on interpolation [3], interpolation properties have been recognized as important properties of logical systems. Recall that a logic L has *interpolation* if whenever $A \to B$ holds in L there exists a formula I in the common language of A and B such that $A \to I$ and $I \to B$ hold in L.

Propositional interpolation properties can be determined and classified with relative ease using the ground-breaking results of Maksimova cf. [7, 6, 5]. This approach is based on an algebraic analysis of the logic in question. In contrast first-order interpolation properties are notoriously hard to determine, even for logics where propositional interpolation is more or less obvious. For example it is unknown whether $\mathrm{G}^{\mathrm{QF}}_{[0,1]}$ (first-order infinitely-valued Gödel logic) interpolates (cf. [1]) and even for $\mathrm{MC}^{\mathrm{QF}}$, the logic of constant domain Kripke frames of three worlds with two top worlds (an extension of MC), interpolation proofs are very hard cf. Ono [8]. This situation is due to the lack of an adequate algebraization of non-classical first-order logics. In this paper we present a proof theoretic methodology to reduce first-order interpolation to propositional interpolation:

$$\left.\begin{array}{r}\text{existence of suitable skolemizations } +\\ \text{existence of Herbrand expansions } +\\ \text{propositional interpolation}\end{array}\right\} \Rightarrow \begin{array}{c}\text{first-order}\\ \text{interpolation.}\end{array}$$

The construction of the first-order interpolant from the propositional interpolant follows this procedure:

1. Develop a validity equivalent skolemization replacing all strong quantifiers[3] in the valid formula $A \to B$ to obtain the valid formula $A_1 \to B_1$.
2. Construct a valid Herbrand expansion $A_2 \to B_2$ for $A_1 \to B_1$. Occurrences of $\exists x B(x)$ and $\forall x A(x)$ are replaced by suitable finite disjunctions $\bigvee B(t_i)$ and conjunctions $\bigwedge B(t_i)$, respectively.
3. Interpolate the propositionally valid formula $A_2 \to B_2$ with the propositional interpolant I^*: $A_2 \to I^*$ and $I^* \to B_2$ are propositionally valid.
4. Reintroduce weak quantifiers to obtain valid formulas $A_1 \to I^*$ and $I^* \to B_1$.

* This abstract is based on the publication [2].

[3] Here we are dealing with quantifiers $\forall$ and $\exists$ such that $A(t) \to \exists x A(x)$ and $\forall x A(x) \to A(t)$ hold. This occurrence of quantifiers is called weak, the dual occurrence is called strong.

5. Eliminate all function symbols and constants not in the common language of A_1 and B_1 by introducing suitable quantifiers in I^* (note that no Skolem functions are in the common language, therefore they are eliminated). Let I be the result.
6. I is an interpolant for $A_1 \to B_1$. $A_1 \to I$ and $I \to B_1$ are skolemizations of $A \to I$ and $I \to B$. Therefore I is an interpolant of $A \to B$.

It is decidable if propositional lattice based finitely-values logics admit the interpolation property [2]. Consequently, it is decidable if finitely-valued first-order logics admit the interpolation property. In this lecture we extend the methodology to prenex fragments of non-classical logics where Skolemization is admissible due to the second epsilon theorem [4].

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