Epistemic Logics of Structured Intensional Groups

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One of the usual assumptions of multi-agent epistemic logic is that groups of agents are given *extensionally* as sets of agents, membership in groups is common knowledge among all agents, and change in membership implies change of identity of a group. This is not how we usually think of groups. We are commonly reasoning in various contexts without knowing groups' extensions—we might routinely refer to groups such as "bot accounts", "democrats", or "correct processes"—and we do not settle for reducing groups to their extensions either, as clearly they can change across the state space of a system, or possible states of the world. Epistemic logics of intensional groups lift the assumptions above, by seeing groups as given to us *intensionally* by a common property that may change its extension from world to world.

In their seminal work [6, 5], Grove and Halpern introduced a multi-agent epistemic logic where groups are labeled by abstract *names* whose extensions can vary from world to world. The language contains two types of modalities: $E_n\varphi$ means that "everyone named n knows that φ ", and $S_n\varphi$ means that "someone named n knows that φ ". They further consider a natural extension of the basic framework where names are replaced by formulas expressing *structured* group-defining concepts. Motivated mainly by applications such as dynamic networks of processes, another framework where the agent set can vary from state to state, have been developed in a form of term-modal logic. Introduced by [4], it builds upon first order logic, indexing modalities by terms that can be quantified over. Epistemic logic with names of [6] was in a sense seminal to the development of term-modal logic, and can be seen as its simple decidable fragment (a closely related language of implicitly quantified modal logic was studied in [10]).

Grove and Halpern's work is enjoying a recent resurgence of interest in the epistemic logic community. [2] considers expansions with non-rigid versions of common and distributed knowledge. Humml and Schröder [8] generalize Grove and Halpern's approach to structured names represented by formulas defining group membership, including e.g. formulas of the description logic ALC. Their abstract-group epistemic logic (AGEL) contains a common knowledge modality as the only modality and, unlike in [2, 6], their group names are rigid.

In this paper, we adopt the perspective that both "everyone labeled a knows" and "someone labeled a knows" modalities form a minimal epistemic language for group knowledge where groups are understood intensionally, and that their labels reflect their structured nature. We use languages built on top of classical propositional language containing modalities $[a], \langle a]$ indexed by elements of an algebra of a given signature of interest. As the main contribution, we set up a general framework for epistemic logics for structured groups in terms of relational semantics involving an algebra of group labels to index (sets of) relations in each world, show how some related logics can be modelled in such a way, generalize relational frames in terms of two-sorted algebras involving propositions and groups, develop an algebraic duality and prove completeness of the minimal logic. The semantics can be seen as an interesting version of monotone neighborhood frame semantics. We further discuss several examples of algebraic signatures giving rise to interesting and useful variants of group structure.

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Frame semantics for structured groups Let Σ be an algebraic similarity type. A Σ-algebra is any $\mathbf{X} = (X, \{o^{\mathbf{X}} \mid o \in \Sigma\})$. A *relational Σ-frame* is $\mathfrak{F} = (W, R, \mathbf{G})$, where $W \neq \emptyset$ ("worlds"); $R \subseteq 2^{W \times W}$ ("agent relations"—the set of available agents); and $\mathbf{G}$ is a Σ-algebra with universe $G \subseteq (2^R)^W$ ("group intensions"). Sets $f(w) \subseteq R$ corresponding to sets of agents can be seen as *intensions* of properties of agents, representing the set of agents that possess the given property in w. Crucially, properties may change their extensions from world to world.

Let Pr, Gr be denumerable sets of propositional variables and group variables respectively. For each Σ, the Σ-*language* is two-sorted, the set of Σ-terms Tm_Σ, and the set of Σ-formulas Fm_Σ are defined by the following grammar:

$$Tm_\Sigma: \quad \alpha := \mathtt{a} \in Gr \mid o(\alpha_1, \ldots, \alpha_n) \qquad Fm_\Sigma: \quad \varphi := \mathtt{p} \in Pr \mid \neg\varphi \mid \varphi \wedge \varphi \mid [\alpha]\,\varphi \mid \langle\alpha]\,\varphi.$$

Formulas $[\alpha]\,\varphi$ read as "Everyone in the group (given by) α knows that φ" and $\langle\alpha]\,\varphi$ read as "Someone in the group (given by) α knows that φ". The *complex algebra* of a relational Σ-frame $\mathfrak{F}$ is $\mathfrak{F}^+ = (\mathbf{F}, \mathbf{G}, [\,]^+, \langle\,]^+)$ where $\mathbf{F}$ is the Boolean algebra of (all) subsets of W; $[\,]^+$ and $\langle\,]^+$ are functions of the type $2^W \times \mathbf{G} \to 2^W$ such that for $a \in G$ and $P \subseteq W$:

$$[a]^+P = \{w \mid \forall r \in a(w) : r(w) \subseteq P\} \qquad \langle a]^+P = \{w \mid \exists r \in a(w) : r(w) \subseteq P\}$$

(where $r(w) = \{u \mid (w,u) \in r\}$). A *model* based on a Σ-frame $\mathfrak{F}$ is $\mathfrak{M} = (\mathfrak{F}, [\![\,]\!])$, where $[\![\,]\!]$ (the "interpretation function") is a homomorphism from $Tm_\Sigma \cup Fm_\Sigma$ to $\mathfrak{F}^+$, that is,

- $[\![o(\alpha_1, \ldots, \alpha_n)]\!] = o^{\mathbf{G}}([\![\alpha_1]\!], \ldots, [\![\alpha_n]\!])$;
- $[\![\neg\varphi]\!] = W \setminus [\![\varphi]\!]$, $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$, $[\![[\alpha]\,\varphi]\!] = [[\![\alpha]\!]]^+[\![\varphi]\!]$, $[\![\langle\alpha]\,\varphi]\!] = \langle[\![\alpha]\!]]^+[\![\varphi]\!]$.

A formula φ is *valid* in a model $\mathfrak{M}$ iff $[\![\varphi]\!]_{\mathfrak{M}} = W_{\mathfrak{M}}$, and valid in a class of frames iff it is valid in each model based on a frame in the class. $Log(\mathsf{K})$ is the set of formulas valid in all frames in K.

Example 1. Epistemic logic with names [2]: Let N ("names"), A ("agents") and W ("worlds") be three non-empty sets. A *relational frame* is (W, A, N, Q, μ), where $Q : A \to 2^{W \times W}$ and $\mu : N \to (W \to 2^A)$. Each relational frame gives rise to a relational $\emptyset$-frame where $R = \{Q_i \mid i \in A\}$ and $G = \{\mu^\#(n) \mid n \in N\}$, where $\mu^\#(n)(w) = \{Q_i \mid i \in \mu(n)(w)\}$. Conversely, every relational $\emptyset$-frame can be seen as a relational frame where $A = R$, Q is the identity function on A, $N = G$ and $\mu(g)(w) = G(w)$. Grove and Halpern [6] consider a version of their framework where groups are referred to by means of formulas of a Boolean language. A simplified version of this framework can be presented as an extension of the relational frames above, if we require that N is a term algebra over terms in the signature $\Sigma_{\mathsf{BA}} = \{\bar{\ }, \wedge, \vee\}$, and that μ satisfies the following conditions (we use n, m as variables ranging over Σ_{BA}-term to highlight the relation to Grove and Halpern's framework):

$$\mu(\bar{n}, w) = W \setminus \mu(n, w) \qquad \mu(n \wedge m, w) = \mu(n, w) \cap \mu(m, w) \qquad \mu(n \vee m, w) = \mu(n, w) \cup \mu(m, w)\,.$$

Every relational frame of this kind gives rise to a relational Σ_{BA}-frame. Conversely, every relational Σ_{BA}-*model* gives rise to a Boolean relational model: $A = R$, Q is the identity function on A, N is the term algebra over Σ_{BA}-terms and $\mu(n) = [\![n]\!]$.

Example 2. Humml and Schröder [8] consider a rigid common knowledge operator labeled by formulas in a fixed agent language $\mathscr{L}_{Ag}$ over a fixed set Ag of agents, defining groups of agents by semantical means of an agent model A. An AGEL frame is $(W, A, \sim)$ with a set $\sim$ of agent relations. The agent language $\mathscr{L}_{Ag}$ determines a signature Σ, and the complex algebra $\mathbf{A}$ of the agent model A is a Σ-algebra (it is the algebra on group propositions $\{[\![\alpha]\!]_A \subseteq Ag \mid \alpha \in \mathscr{L}_{Ag}\}$). As the agent language conservatively extends classical propositional logic, this algebra carries a boolean structure. It gives rise to a Σ-relational frame where $R = \sim$ and $\mathbf{G}$ is determined by $\mathbf{A}$ as $G = \{\sim_{[\![\alpha]\!]_A} \mid \alpha \in \mathscr{L}_{Ag}\}$ where $\sim_{[\![\alpha]\!]_A}$ is the union of relations of agents satisfying α (and $G(w)$ is constant along all possible worlds).

Logic An *epistemic logic with structured intensional groups over* Σ (a Σ-*logic*) is any set $L \subseteq Fm_\Sigma$ such that (for all $\alpha \in Tm_\Sigma$)

1. L contains all substitution instances of classical tautologies and is closed under Modus Ponens;
2. L contains all formulas of the form (K) $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$ and is closed under the Necessitation rule (Nec) $\dfrac{\varphi}{[\alpha]\varphi}$;
3. L contains all formulas of the form $\neg[\alpha]\bot \to \langle\alpha]\top$ and $\langle\alpha]\varphi \wedge [\alpha]\psi \to \langle\alpha](\varphi \wedge \psi)$.

Theorem 1 (Completeness)**.** *The smallest* Σ*-logic is the set of* Σ*-formulas valid in all relational* Σ*-models.*

The relational Σ-frames can be alternatively seen as monotone neighborhood Σ-frames, if we understand sets $\{r(w) \mid r \in a(w)\}$ as core neighborhood sets [7, 9]. A similar perspective has recently been adopted in [2], and also by [3] on a somebody-knows modality, previously studied by [1]. Neither of the approaches in [7, 3] however includes both $\exists\forall$ and $\forall\forall$ types of modalities. The algebraic structure underlying the labelling of groups needs to be captured additionally (e.g. morphisms of neighborhood Σ frames additionally involve an algebraic homomorphism $g : \mathbf{G} \to \mathbf{G}'$ which can be interpreted as allowing to 'rename' the groups along frame morphisms in a structured way). The categories of relational Σ-frames and neighborhood Σ-frames are equivalent.

Example 3 (JS-logic and distributed knowledge)**.** One of the simplest forms of structure imposed on groups of agents corresponds to taking unions of sets of agents. It is modelled by a *semilattice* structure on the set of intensional groups, where the neutral element is an "inconsistent" intensional group that has an empty extension in each world. A *relational join-semilattice frame* (relational js-frame) is a relational Σ_{SL}-frame where $0^{\mathbf{G}}(w) = \emptyset$ and $(f +^{\mathbf{G}} g)(w) = f(w) \cup g(w)$. The *join-semilattice logic* is the smallest Σ_{SL}-logic that contains all formulas of the following forms:

$$\top \to [0]\varphi \qquad (1) \qquad\qquad [\alpha+\beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi \qquad (3)$$

$$\langle 0]\varphi \to \bot \qquad (2) \qquad\qquad \langle\alpha+\beta]\varphi \leftrightarrow \langle\alpha]\varphi \vee \langle\beta]\varphi \qquad (4)$$

In the extensional setting, φ is distributed knowledge in a group iff it is satisfied in every world accessible using the intersection of the relations in the group. The intersection of each non-empty subset of a set of relations-agents X gives rise to a new relation-agent X', distributed knowledge in X then corresponds to the "somebody knows" operator applied to X'. The structure induced by distributed knowledge in the intensional setting is that of *relational closure js-frame*, where $^{\cap}$ is a unary closure operator on G, and $f^{\cap^{\mathbf{G}}}(w) = \{r \in R \mid r(w) = \bigcap_{r_i \in X} r_i(w)$ for some $\emptyset \neq X \subseteq f(w)\}$.

Further and on-going work Information about *meta-beliefs* ("i believes that j believes that φ") is crucial to many multi-agent scenarios. The question what is a reasonable notion of composition of intensional groups (intensional sets of relations), is not immediate to answer. We have so far considered an interesting version of *intensional composition*, which gives rise to an algebraic structure of *right-unital magmas* ($(M, \cdot, 1)$ where $\cdot$ is a binary operation on M and $1 \in M$ such that $x \cdot 1 = x$ for all $x \in M$). Once we have a working notion of composition, we may use the standard fixpoint construction to introduce *common knowledge*. An additional topic for future work is the exploration of variants of the notion of intensional composition. In particular, we are interested if there is a variant giving rise to a monoid structure on intensional groups.

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