On Splittings in Lattices of Quasivarieties

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Abstract

We show that similarly to splittings in lattices of varieties, splitting algebras in lattices of quasivarieties do not have to be finitely presented.

1 Introduction

Initially, splittings in the lattices of extensions of logics (especially, intuitionistic propositional logic and some modal logics viewed as sets of formulas closed under rules of Modus Ponens and substitution) were used for studying the properties of these lattices, for instance, for proving that the cardinality is not countable (for more information, see, e.g., [6, 7]). As splitting formulas are &-irreducible, they were used for describing the logics having irredundant axiomatization. It turned out that not every (even intermediate) logic can be axiomatized by &-irreducible formulas, and this led to modification of &-irreducible formulas to Zakharyaschev's canonical formulas. It also turned out (see, e.g. [3]) that splitting formulas can be used as anti-axioms in the refutation systems. The algebraic counterparts of the logics admitting some kind of the Deduction Theorem are varieties with EDPC. If we view the logics as structural consequence relations, the quasivarieties become the algebraic counterparts, and the questions asked about splittings in lattices of varieties can be asked about splittings in lattices of quasivarieties.

One of such questions was whether every splitting algebra in a variety with EDPC is finitely presented. The counterexamples for the varieties of modal and Heyting algebras were given in [6, 2]. In Section 3, we give a counterexample by exhibiting a quasivariety of Heyting algebras which contains a splitting algebra that is not finitely presented. Since in the case when (quasi)variety is not generated by its finite members and splitting algebra is not unique (up to isomorphism), the problem of finite presentability of splitting algebra should be rephrased in the following way:

Problem 1.1. Can every splitting pair in a lattice of all sub(quasi)varieties of a (quasi)variety be defined by a finitely presented splitting algebra?

2 Basic Facts

All algebras are assumed to be of the same arbitrary but fixed type. As usual, for a class of algebras $\mathcal{K}$, we take: $\mathbb{H}\mathcal{K}, \mathbb{S}\mathcal{K}, \mathbb{P}\mathcal{K}, \mathbb{P}_u\mathcal{K}$ to be respectively the classes of all homomorphic images, all isomorphic copies of all subalgebras, all isomorphic images of all direct products and ultrapoducts of members of $\mathcal{K}$. We also let $\mathbb{N}\mathbf{A}$ be a class of all algebras $\mathbf{B}$ such that $\mathbf{A} \notin \mathbb{S}\mathbf{B}$.

If $\mathcal{Q}$ is a quasivariety and $\mathbf{A} \in \mathcal{Q}$, a congruence θ of $\mathbf{A}$ is a ***$\mathcal{Q}$-congruence*** if $\mathbf{A}/\theta \in \mathcal{Q}$; in addition, $\mathbf{A}$ is $\mathcal{Q}-$ ***irreducible*** if the meet of all distinct from identity $\mathcal{Q}$-congruences of $\mathbf{A}$ is distinct from identity – ***$\mathcal{Q}$-monolith***. The identity congruence is denoted by $\varepsilon_{\mathbf{A}}$.

To prove the main theorem, we need the following facts from the theory of quasivarieties.

Proposition 2.1 ([5, Theorem 1.2.8.]). *If an algebra* $\mathbf{A}$ *is locally embeddable into a class* $\mathcal{K}$*, then* $\mathbf{A}$ *is embeddable into an ultraproduct of some algebras in* K*, that is,* $\mathbf{A} \in \mathbb{S}\mathbb{P}_u\mathcal{K}$.

Because quasivarieties are closed under $\mathbb{P}_u$ and $\mathbb{S}$, the following holds.

Corollary 2.2. *If algebra* $\mathbf{A}$ *is locally embeddable in a quasivariety* $\mathcal{Q}$*, then* $\mathbf{A} \in \mathcal{Q}$*.*

Proposition 2.3. *Let* $\mathcal{Q} = \mathsf{Q}(\mathcal{K})$ *be a quasivariety generated by a class of algebras* $\mathcal{K}$ *and* $\mathbf{A}$ *be a* $\mathcal{Q}$*-irreducible* $\mathcal{Q}$*-finitely presentable algebra. Then,* $\mathbf{A} \in \mathbb{S}\mathcal{K}$*.*

Let $\Lambda_q(\mathcal{Q})$ be a lattice of all subquasivarieties of quasivariety $\mathcal{Q}$. A pair $(\mathcal{Q}_1, \mathcal{Q}_2)$, where $\mathcal{Q}_1, \mathcal{Q}_2 \in \Lambda_q(\mathcal{Q})$ is a ***splitting pair*** in $\Lambda_q(\mathcal{Q})$ if $\mathcal{Q}_1 \not\subseteq \mathcal{Q}_2$ and for every $\mathcal{Q}' \in \Lambda_q(\mathcal{Q})$,

$$\mathcal{Q}_1 \subseteq \mathcal{Q}' \text{ or } \mathcal{Q}' \subseteq \mathcal{Q}_2. \tag{Split}$$

Let us observe that if $(\mathcal{Q}_1, \mathcal{Q}_2)$ is a splitting pair, immediately from (Split) it follows that $\mathcal{Q}_2$ is uniquely defined by $\mathcal{Q}_1$ (and vice-versa).

The $\mathcal{Q}$-irreducible finitely-generated algebras which define splitting pair in $\mathcal{Q}$ are called ***$\mathcal{Q}$-splitting algebras*** (and we omit the reference to quasivariety if no confusion arises).

Similarly to lattices of varieties, every splitting $(\mathcal{Q}_1, \mathcal{Q}_2)$ is defined by a finitely generated $\mathcal{Q}$-irreducible algebra $\mathbf{A}$, namely, $\mathcal{Q}_1 = \mathsf{Q}(\mathbf{A})$, and $\mathcal{Q}_2$ is the largest quasivariety from $\Lambda_q(\mathcal{Q})$ not containing $\mathbf{A}$, which will be denoted by $\overline{\mathsf{Q}}(\mathbf{A})$. Thus, in every splitting pair $(\mathcal{Q}_1, \mathcal{Q}_2)$,

$$\mathcal{Q}_1 = \mathsf{Q}(\mathbf{A}), \text{ and } \mathcal{Q}_2 = \overline{\mathsf{Q}}(\mathbf{A}).$$

Proposition 2.4. *Let* $\mathcal{Q}$ *be a quasivariety and* $\mathbf{A}$ *be a* $\mathcal{Q}$*-splitting algebra. Then,*

(a) $\overline{\mathsf{Q}}(\mathbf{A}) \subseteq \mathbb{N}\mathbf{A}$ *and hence,* $\overline{\mathsf{Q}}(\mathbf{A}) \subseteq \mathbb{N}\mathbf{A} \cap \mathcal{Q}$*;*

(b) if $\mathbf{A} \in \mathcal{Q}$ *is* $\mathcal{Q}$*-irreducible algebra and it is finitely presented in* $\mathcal{Q}$ *, then class* $\mathbb{N}\mathbf{A}$ *is a quasivariety and consequently,* $\mathbb{N}\mathbf{A} \cap \mathcal{Q} = \overline{\mathsf{Q}}(\mathbf{A})$*.*

(a) immediately follows from the observation that $\mathbf{A} \notin \overline{\mathsf{Q}}(\mathbf{A})$; for (b), see [1, Corollary 3].

Corollary 2.5. *In any quasivariety* $\mathcal{Q}$*, every finitely-presented and* $\mathcal{Q}$*-irreducible algebra* $\mathbf{A}$ *is a* $\mathcal{Q}$*-splitting algebra and* $\overline{\mathsf{Q}}(\mathbf{A}) = \mathbb{N}\mathbf{A}$*. Thus, every finite* $\mathcal{Q}$*-irreducible algebra of a finite type is a* $\mathcal{Q}$*-splitting algebra.*

3 Main Theorem

We consider Heyting algebras in signature $\Sigma = \{\to, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1}\}$, and variety of all Heyting algebras is denoted by $\mathcal{H}$. Let $\mathcal{Q}_1$ be the quasivariety generated by some one-generated free Heyting algebra $\mathbf{F}_1 := \mathbf{F}_{\mathcal{H}}(1)$, and we let $\mathbf{F}_1'$ be a Heyting algebra obtained from $\mathbf{F}_1$ by adding a new top element (see Fig. 1, where element g is a free generator of $\mathbf{F}_1$).

Our goal is to prove the following theorem.

Theorem 3.1. *The following holds:*

(a) $\mathbf{F}_1$ *is a splitting algebra in* $\mathcal{Q}_1$*;*

(b) $\mathbf{F}_1'$ *is subdirectly irreducible and it generates* $\mathcal{Q}_1$*, hence,* $\mathbf{F}_1'$ *is a splitting algebra in* $\mathcal{Q}_1$*;*

(c) $\mathbf{F}_1'$ *is not finitely presented in* $\mathcal{Q}_1$*.*

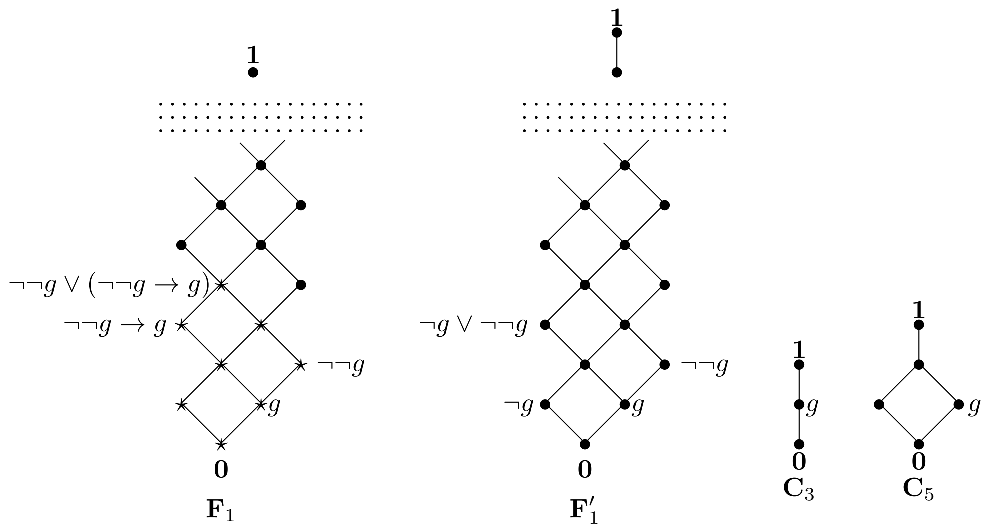

Figure 1: Splitting Heyting algebras

Proof of (a). Since $\mathbf{F}_1$ is a free algebra in $\mathcal{H}$, it is free in $\mathcal{Q}_1$ as well and hence, it is finitely presented in $\mathcal{Q}_1$. If we show that $\mathbf{F}_1$ is $\mathcal{Q}_1$-irreducible, we can apply Corollary 2.5, and complete the proof of (a).

To prove that $\mathbf{F}_1$ is $\mathcal{Q}_1$-irreducible, we will demonstrate that it has a nontrivial $\mathcal{Q}_1$-monolith, namely, we will show that $\mathbf{F}_1$ contains distinct from $\mathbf{1}$ elements which cannot be distinguished from $\mathbf{1}$ by any $\mathcal{Q}_1$-congruence; more precisely, we will show that $\neg\neg g \vee (\neg\neg g \to g)$ belongs to the monolith and it is its smallest element.

Let $g' := \neg\neg g \vee (\neg\neg g \to g)$ and let us consider congruence $\theta := \theta(g', \mathbf{1})$. First, let us observe that θ is a $\mathcal{Q}_1$-congruence. Indeed, $\mathbf{F}_1/\theta$ is an 8-element one-generated Heyting algebra (elements of which are denoted by $\star$), which is a subdirect product of three- and five-element algebras $\mathbf{C}_3$ and $\mathbf{C}_5$, the Hasse diagrams of which are depicted in Fig. 1. Algebras $\mathbf{C}_3$ and $\mathbf{C}_5$ are isomorphic to subalgebras of $\mathbf{F}_1$: $\mathbf{C}_3$ is isomorphic to the subalgebra generated by element $\neg\neg g \to g$, and $\mathbf{C}_5$ – by element $\neg\neg g$. Thus, $\mathbf{C}_3, \mathbf{C}_5 \in \mathcal{Q}_1$ and consequently, $\mathbf{F}_1/\theta \in \mathcal{Q}_1$.

Next, let us note that θ is the smallest distinct from identity $\mathcal{Q}_1$-congruence of $\mathbf{F}_1$, because the following quasi-identity holds in $\mathbf{F}_1$, but it does not hold in any quotient algebra by any congruence θ' such that $\varepsilon_{\mathbf{F}_1} \subset \theta' \subset \theta$:

$$((x \to y) \to (x \vee z)) \vee u \approx \mathbf{1} \Rightarrow (((x \to y) \to x) \vee ((x \to y) \to y) \vee u) \approx \mathbf{1}.$$

Indeed, let us recall from [4, Proposition 5.41] that the above quasi-identity holds in a finite Heyting algebra if and only if this algebra is a subdirect product of the projective Heyting algebras; $\mathbf{C}_3, \mathbf{C_5}$ and a two-element Heyting algebra $\mathbf{C}_2$ are the only finite projective one-generated Heyting algebras in $\mathcal{Q}_1$, and $\mathbf{F}_1/\theta'$ is not a subdirect product of $\mathbf{C}_3, \mathbf{C_5}$ and $\mathbf{C}_2$, because identity $\neg\neg x \vee (\neg\neg x \to x) \approx \mathbf{1}$ holds in $\mathbf{C}_2, \mathbf{C}_3, \mathbf{C_5}$ but not in $\mathbf{F}_1/\theta'$.

Proof of (b). To prove (b) we will show that algebra $\mathbf{F}'_1$ is locally embeddable in $\mathbf{F}_1$ and hence, by Corollary 2.2, $\mathbf{F}'_1 \in \mathsf{Q}(\mathbf{F}_1) = \mathcal{Q}_1$. To that end, we will exhibit a class of finite partial subreducts of $\mathbf{F}'_1$ in which every finite partial subreduct of $\mathbf{F}'_1$ can be embedded, and then, we will apply Corollary 2.2.

With each element $a \in \mathbf{F}_1'$ such that $\mathbf{0} < a < \omega$ we associate a partial subreduct

$$\mathbf{F}^a := \mathbf{F}_1'[\{b \in \mathbf{F}_1' \mid b \leq a\} \cup \{o, \mathbf{1}\}, \Sigma]$$
$$\mathbf{F}^o := \mathbf{F}_1'[\{o, \mathbf{1}\}, \Sigma], \quad \mathbf{F}^{\mathbf{1}} := \mathbf{F}_1'[\{\mathbf{1}\}, \Sigma].$$

It is not hard to see that for every finite subset of elements $A \subseteq \mathbf{F}_1'$, there are two cases to consider: (i) A does not contain elements distinct from o and $\mathbf{1}$, and (ii) A contains elements distinct from o and $\mathbf{1}$.

Case (i) is trivial. In Case (ii), because A is finite, one can take a disjunction of all elements from A distinct from o and $\mathbf{1}$, let us denote it by b, and observe that $[F] : b$ is a desired finite partial subreduct.

Let us note that $\mathbf{F}_1$ is isomorphic to a subalgebra of $\mathbf{F}_1'$ generated by element g, so, to simplify the proof, we can assume that $\mathbf{F}_1$ is a subalgebra of $\mathbf{F}_1'$.

First, let us note that $\mathbf{F}^{\mathbf{1}} = \mathbf{F}_1[\{\mathbf{1}\}, \Sigma]$. In addition, subreduct $\mathbf{F}^o$ can be embedded into subreduct $\mathbf{F}_1[\{\mathbf{0}, \mathbf{1}\}, \Sigma]$. Thus, we need to consider only subreducts $\mathbf{F}^b$, where $b < o$.

Suppose that $b < o$. Then we can take a map φ that sends every element of $\mathbf{F}^b$ distinct from o into itself. Now, we need to extend φ to o.

Because $b < o$, there is element $c \in \mathbf{F}_1$ distinct from $\mathbf{1}$ and such that $c \not\leq b$. Let us consider element $d := c \vee (c \to d)$. We leave for the reader to verify that

$$\text{for every } a \leq b, d \to a = a,$$

and therefore, we can extend φ by letting $\varphi(o) = d$, and this completes the proof of (b).

Proof of (c). To prove that algebra $\mathbf{F}_1'$ is not finitely presented in $\mathcal{Q}_1$, we will show that $\mathbf{F}_1'$ can not be embedded into $\mathbf{F}_1$, and then we can apply Proposition 2.3, because $\mathcal{Q}_1 = \mathsf{Q}(\mathbf{F}_1)$ and $\mathbf{F}_1'$ is subdirectly irreducible: the congruence $\theta(o, \mathbf{1})$ is its monolith.

For contradiction: assume that $\varphi : \mathbf{F}_1' \longrightarrow \mathbf{F}_1$ is an embedding.

Let us observe that in $\mathbf{F}_1'$, elements $\mathbf{0}, \neg g, \neg\neg g, \mathbf{1}$ are regular, that is, they satisfy condition $\neg\neg x = x$. In addition, all elements $\{a \in \mathbf{F}_1' \mid g \vee \neg g \leq a\}$ are dense, that is, they satisfy condition $\neg x = \mathbf{0}$. Thus, g is the only element of $\mathbf{F}_1'$ which is neither regular, nor dense. Similarly, in $\mathbf{F}_1$ element g is the only element which is neither regular, nor dense. Hence, $\varphi : g \mapsto g$.

Let us recall that element g generates algebra $\mathbf{F}_1$, hence, φ is a map onto $\mathbf{F}_1$ and consequently, either $\varphi(o) = \mathbf{1}$, and φ is not one-to-one map, or $\varphi(o) < \mathbf{1}$ and there is an element $a \in \mathbf{F}_1$ such that $\varphi(o) = a$, which contradicts that φ is an isomorphism. □

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