Intuitionistic and Analytic Implication in Truthmaker Semantics

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Richard B. Angell (1989) introduced the logic AC of analytic containment. It is a logic that aims at capturing a notion of entailment satisfying the following condition: A entails B only if the meaning of B is contained in the meaning of A. Synonymity is then regarded as analytic equivalence definable in terms of this notion of entailment: A is synonymous with B if and only if A entails B and B entails A.

This notion of entailment is formalized as an implication for which we will use the symbol $\Rightarrow$. In Angell's system this implication can occur only as the main connective and cannot be embedded under other operators. The logic AC has some non-classical features. In fact, it is a subsystem of the logic known as first degree entailment (FDE). As such, it has the variable sharing property. It even possesses a stronger version of the variable sharing property that more common relevant logics (including FDE) lack: $\alpha \Rightarrow \beta$ is a theorem of AC only if α contains all the propositional variables occurring in β (proven in (Ferguson, 2016) and (Fine, 2016)). As a consequence, AC does not validate such principles as $p \Rightarrow (p \vee q)$. This makes perfect sense given the intended interpretation of implication (the proposition represented by $p \vee q$ includes extra content that is not contained in p).

Angell actually did not use analytic implication as a primitive connective. Instead, he took as primitive synonymity, or analytic equivalence, which we will denote by $\Leftrightarrow$. Analytic implication can be defined in terms of analytic equivalence as follows: $\alpha \Rightarrow \beta =_{def} \alpha \Leftrightarrow (\alpha \wedge \beta)$.

AC was originally introduced only as a syntactic system, with the informal interpretation sketched above but without any precisely defined formal semantics. Much later, Kit Fine (2016) formulated an interesting and adequate truthmaker semantics for AC which corresponds nicely to the original intended informal interpretation. Thus far, no extensions of AC have been introduced that are defined over a language allowing for higher degree formulas (i.e. formulas including nested conditionals).

In our paper, we expand Angell's analytic implication and transform it into a proper object language connective that can be arbitrarily embedded into more complex formulas. This is achieved by a definition of analytic implication in terms of two other connectives. The first one is intuitionistic implication, with its truthmaker semantic characterization also provided by Fine (2016). The other one is rather unusual and it is obtained by a semantic symmetry from the semantic characterization of the intuitionistic implication. The resulting framework allows us to define Angell's analytic containment and analytic equivalence. By this approach we obtain a logic AC^+ that conservatively extends both AC and positive intuitionistic logic.

The logic AC^+ is determined only semantically. The problem of providing an axiomatic characterization is left for future research. But we will axiomatize a related logic AC^{++} that differs from AC^+ in that it treats atomic formulas as having a unique truthmaker. AC^{++} also conservatively extends AC but in the other direction it does not conservatively extend positive intuitionistic logic but rather its "inquisitive variant" (Punčochář, 2016; Ciardelli, Iemhoff & Yang, 2020).

In the rest of this abstract, let us formulate more precisely the main definitions and results. We first formulate Fine's truthmaker semantics for AC combined with his treatment of intuitionistic logic. We say that a partial order $\langle S, \sqsubseteq \rangle$ is *complete* if every $T \subseteq S$ has the least upper bound $\bigsqcup T \in S$. In each complete partial order there is also the greatest lower bound $\bigsqcap T$ for each $T \subseteq S$ that can be defined as the least upper bound of all lower bounds of T. As usual, we write $s \sqcup t$ and $s \sqcap t$ instead of $\bigsqcup\{s,t\}$ and $\bigsqcap\{s,t\}$. Each complete partial order has the least element $0 = \bigsqcup \varnothing$, and the greatest element $1 = \bigsqcap \varnothing$.

Let $\langle S, \sqsubseteq \rangle$ be a complete partial order. Let $s \Rightarrow t = \bigsqcap\{u \in S \mid t \sqsubseteq u \sqcup s\}$, for each $s, t \in S$. We say that $\langle S, \sqsubseteq \rangle$ is *residuated* if $t \sqsubseteq s \sqcup (s \Rightarrow t)$, for every $s, t \in S$. An *E-frame* (exact frame) is a residuated complete partial order. E-frames correspond exactly to complete Brouwer algebras, i.e. complete Heyting algebras turned upside down. For each E-frame $\langle S, \sqsubseteq \rangle$ we define two orderings, $\preccurlyeq$ and $\precsim$, on the power set of S. Let $T, U \subseteq S$. Then

$$U \preccurlyeq T \text{ iff } \forall t \in T\ \exists u \in U : u \sqsubseteq t, \text{ and } U \precsim T \text{ iff } \forall u \in U\ \exists t \in T : u \sqsubseteq t.$$

We say that T *contains* U if $U \preccurlyeq T$ and $U \precsim T$. If T contains U and U contains T then we say that T and U are *analytically equivalent*.

The language L_1 is built up from atomic formulas by negation $\neg$, conjunction $\wedge$, and disjunction $\vee$. An *E-model* is a tuple $\langle S, \sqsubseteq, V^+, V^- \rangle$, where $\langle S, \sqsubseteq \rangle$ is an E-frame and both V^+ and V^- are functions assigning to each atomic formula a non-empty subset of S. The elements of S in an E-model can be called *situations*. In accordance with Fine's truthmaker semantics, we define an exact truthmaking relation $\Vdash^+$ and an exact falsemaking relation $\Vdash^-$ relating situations in S and L_1-formulas. The relations are defined recursively as follows: $s \Vdash^+ p$ iff $s \in V^+(p)$; $s \Vdash^- p$ iff $s \in V^-(p)$; $s \Vdash^+ \neg\alpha$ iff $s \Vdash^- \alpha$; $s \Vdash^- \neg\alpha$ iff $s \Vdash^+ \alpha$; $s \Vdash^+ \alpha \wedge \beta$ iff $\exists t, u \in S$: $t \Vdash^+ \alpha$, $u \Vdash^+ \beta$ and $s = t \sqcup u$; $s \Vdash^- \alpha \wedge \beta$ iff $s \Vdash^- \alpha$ or $s \Vdash^- \beta$; $s \Vdash^+ \alpha \vee \beta$ iff $s \Vdash^+ \alpha$ or $s \Vdash^+ \beta$; $s \Vdash^- \alpha \vee \beta$ iff $\exists t, u \in S$: $t \Vdash^- \alpha$, $u \Vdash^- \beta$ and $s = t \sqcup u$.

Let $[\alpha]_{\mathcal{M}}$ denote the set of all truthmakers of α in a given E-model $\mathcal{M}$, that is $[\alpha]_{\mathcal{M}} = \{s \in S \mid s \Vdash^+ \alpha \text{ in } \mathcal{M}\}$. The subscript $\mathcal{M}$ will be omitted if no confusion arises.

An equivalential L_1-formula (or L_1^e-formula, for short) is an expression of the form $\alpha \Leftrightarrow \beta$ where α and β are L_1-formulas. We say that an L_1^e-formula $\alpha \Leftrightarrow \beta$ *holds in an E-model* if $[\alpha]$ and $[\beta]$ are analytically equivalent. The following result is from (Fine, 2016).

Theorem 1. *Let $\alpha \Leftrightarrow \beta$ be an L_1^e-formula. $\alpha \Leftrightarrow \beta$ holds in all E-models if and only if $\alpha \Leftrightarrow \beta$ is derivable in the Angell's system* AC.

Now we extend the language L_1 with two new binary connectives $\rightharpoondown$ and $\rightharpoonup$, and the resulting language will be denoted as L_2. We can define two additional connectives:

$$\varphi \Rightarrow \psi =_{def} (\varphi \rightharpoondown \psi) \wedge (\varphi \rightharpoonup \psi) \text{ and } \varphi \Leftrightarrow \psi =_{def} (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi).$$

In the language L_2, the semantic clauses for atomic formulas, negation, conjunction and disjunction are as before. We equip $\rightharpoondown$ with Fine's (2014) positive clause for intuitionistic implication and $\rightharpoonup$ with a symmetric clause. There are more reasonable options how to set the negative clauses for these operators. Here we formulate just one of these options:

- $s \Vdash^+ \varphi \rightharpoondown \psi$ iff $\exists f : [\varphi] \to [\psi]$ such that $s = \bigsqcup\{t \Rightarrow f(t) \mid t \in [\varphi]\}$,

- $s \Vdash^- \varphi \rightharpoondown \psi$ iff $\exists f : [\varphi] \to [\neg\psi]$ such that $s = \bigsqcup\{t \sqcup f(t) \mid t \in [\varphi]\}$,
- $s \Vdash^+ \varphi \rightharpoonup \psi$ iff $\exists f : [\psi] \to [\varphi]$ such that $s = \bigsqcup\{f(t) \Rightarrow t \mid t \in [\psi]\}$,
- $s \Vdash^- \varphi \rightharpoonup \psi$ iff $\exists f : [\neg\psi] \to [\varphi]$ such that $s = \bigsqcup\{f(t) \sqcup t \mid t \in [\neg\psi]\}$.

Let φ be an L_2-formula. We say that φ *holds in an E-model* $\mathcal{M}$ if $0 \Vdash^+ \varphi$ in $\mathcal{M}$. We say that φ *is valid in* AC^+ if φ holds in every E-model. The following result follows from (Fine, 2014).

Theorem 2. *The* $\{\neg, \rightharpoonup\}$*-free fragment of* AC^+ *is identical with the positive fragment of intuitionistic logic (where* $\rightharpoondown$ *is the intuitionistic implication).*

We will show the following connection between AC^+ and AC.

Theorem 3. AC^+ *conservatively extends* AC*.*

As intended, in contrast to the system AC, where $\Rightarrow$ and $\Leftrightarrow$ can occur only as the main connectives, in AC^+ they can be arbitrarily embedded.

An axiomatic characterization of AC^+ is an open problem which is left for future research. However, we will characterize by a deductive system a logic closely related to AC^+. It is an extension of AC^+ that we will call AC^{++}.

We say that an E-model is *simple* if for every atomic formula p, both $V^+(p)$ and $V^-(p)$ are singletons. We say that an L_2-formula φ *is valid in* AC^{++} if φ holds in every simple E-model. We can prove the following result:

Theorem 4. AC^{++} *conservatively extends* AC*. Moreover, the* $\{\neg, \rightharpoonup\}$*-free fragment of* AC^{++} *is identical with the positive fragment of inquisitive intuitionistic logic from (Punčochář, 2016).*

AC^{++} can be axiomatized as positive intuitionistic logic equipped with several additional axioms determining how negation distributes over other operators, and extended with the following specific axioms in which $\varphi \leftharpoondown\!\!\rightharpoondown \psi$ is a shorthand for $(\varphi \rightharpoondown \psi) \wedge (\psi \rightharpoondown \varphi)$ and α and β range over formulas built up from literals using only $\wedge, \rightharpoondown, \rightharpoonup$.

$$\text{A1 } (\alpha \rightharpoondown \beta) \leftharpoondown\!\!\rightharpoondown (\alpha \rightharpoonup \beta) \quad \text{A2 } (\chi \rightharpoonup (\varphi \vee \psi)) \leftharpoondown\!\!\rightharpoondown ((\chi \rightharpoonup \varphi) \wedge (\chi \rightharpoonup \psi))$$

$$\text{A3 } ((\varphi \vee \psi) \rightharpoonup \alpha) \leftharpoondown\!\!\rightharpoondown ((\varphi \rightharpoonup \alpha) \vee (\psi \rightharpoonup \alpha)) \quad \text{A4 } (\alpha \rightharpoondown (\varphi \vee \psi)) \leftharpoondown\!\!\rightharpoondown ((\alpha \rightharpoondown \varphi) \vee (\alpha \rightharpoondown \psi))$$

As one might expect, we lose the variable sharing property in AC^{++}. For example $q \Rightarrow (p \rightharpoondown p)$ is valid in AC^{++}. Nevertheless, the interpretation of $\Rightarrow$ as containment still makes sense if we accept that logical truths have no content, or more precisely a null content that is included in any other content. The first degree fragment, i.e. AC, has the variable sharing property because there are no implication-free valid formulas.

References

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