Correspondence Theory for Modal Logic with Counting ML(#)

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We consider the frame correspondence theory (in the sense of [1, Section 3]) for modal logic with counting ML(#) introduced in [5, Section 7], with respect to image-finite Kripke frames (where each node has finitely many successors). The basic proof strategy is to make use of the characterization results for numerical definability of shallow formulas (see Section 2), and transform the correspondence problem of ML(#) to the correspondence problem of graded modal logic.

1 Modal Logic with Counting ML(#)

We briefly summarize the language and semantics for modal logic with counting ML(#) in [5]. The formulas of the language for modal logic with counting ML(#) is defined as follows:

$$\begin{aligned} \text{formulas: } & p \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \#\varphi \succsim \#\psi \\ \text{numerical terms: } & \#\varphi \end{aligned}$$

ML(#)-formulas are interpreted on image-finite Kripke frames $\mathbb{F} = (W, R)$ and Kripke models $\mathbb{M} = (\mathbb{F}, V)$. We use $R_s = \{t : Rst\}$ as the set of successors of s. The satisfaction relation for the basic case and Boolean connectives are defined as usual. For numerical terms, $[\![\#\varphi]\!]^{\mathbb{M},s} = |R_s \cap [\![\varphi]\!]^{\mathbb{M}}|$, i.e. $[\![\#\varphi]\!]^{\mathbb{M},s}$ is the number of successors of s where φ is true. For cardinality comparison formulas, $\mathbb{M}, s \Vdash \#\varphi \succsim \#\psi$ iff $[\![\#\varphi]\!]^{\mathbb{M},s} \geq [\![\#\psi]\!]^{\mathbb{M},s}$. It is easy to see that the standard modality $\Diamond\varphi$ can be defined as $\neg(\#\bot \succsim \#\varphi)$. A formula is shallow if it is a Boolean combination of formulas of the form $\#\varphi \succsim \#\psi$, where φ, ψ are classical propositional formulas.

For the convenience of the correspondence algorithm, we consider an expanded language for second-order modal logic with counting SOML(#), where we introduce existential quantified formulas of the form $\exists p\varphi$ (where p is a propositional variable) to represent for the existence of a valuation of p such that the ML(#)-formula φ is satisfied under this valuation. Given any $X \subseteq W$, we use V^p_X denote a valuation which is the almost same as V except that $V^p_X(p) = X$. The satisfaction relation clause for $\exists p\varphi$ is defined as follows: $\mathbb{M}, w \Vdash \exists p\varphi$ iff there exists an $X \subseteq W$ such that $(W, R, V^p_X), w \Vdash \varphi$.

2 The Power of Definability of ML(#)

In this section, we use $\mathbb{N}$ to denote the set of natural numbers.

Definition 1. *Given any $X \subseteq \mathbb{N}$, a shallow ML(#)-formula φ with all propositional variables occurring in $\boldsymbol{p}$ defines X if for all image-finite pointed Kripke frames $(\mathbb{F}, s)$,*

$$(\mathbb{F}, s) \Vdash \exists \boldsymbol{p}\varphi \text{ iff } |R_s| \in X.$$

Definition 2 (Semilinear sets). *A subset $X \subseteq \mathbb{N}$ is said to be linear if it is of the form $X = \{a + b_1 \cdot x_1 + \ldots + b_n \cdot x_n \mid x_1, \ldots, x_n \in \mathbb{N}\}$ for some fixed $a, b_1, \ldots, b_n \in \mathbb{N}$. A subset $X \subseteq \mathbb{N}$ is said to be semilinear if it is a finite union of linear subsets.*

Definition 3. *We say that a subset X of $\mathbb{N}$ is closed under taking multiples, if for any $n \in X$ and $2 \leq m \in \mathbb{N}$, we have that $m \cdot n \in \mathbb{N}$.*

Proposition 1 ([3]). *The subsets definable by shallow ML(#)-formulas are exactly those semilinear sets that are closed under taking multiples.*

Proposition 2. *For any definable subset $X \subseteq \mathbb{N}$ and any ML(#)-formula θ, there is an ML(#)-formula $\#\theta \in X$ such that for all image-finite pointed Kripke models $(\mathbb{M}, s)$,*

$$(\mathbb{M}, s) \Vdash \exists \boldsymbol{p}(\#\theta \in X) \text{ iff } |R_s \cap [\![\theta]\!]^{\mathbb{M}}| \in X,$$

where $\exists \boldsymbol{p}$ quantify over all propositional variables in $\#\theta \in X$ that are not in θ.

3 Correspondence Theory: An Informal Discussion

In correspondence theory for basic modal logic (cf. [1, Section 3]), the standard technique is to use minimal valuations of propositional variables to eliminate them, which depends on the positive and negative occurrences of the propositional variables at certain positions.

Let us first examine the shape of Sahlqvist implications (a fragment of modal formulas which have first-order correspondents):

- We first define positive (resp. negative) formulas as formulas where each propositional variable is in the scope of an even (resp. odd) number of negations (where $\alpha \to \beta$ is regarded as $\neg\alpha \vee \beta$).
- Then we define the Sahlqvist antecedents Ant as formulas built up from $\Box^n p, \bot, \top$ and negative formulas NEG by applying $\vee, \wedge, \Diamond$.
- Finally, we define a Sahlqvist implication as $\mathsf{Ant} \to \mathsf{POS}$ where POS is a positive formula.

For the computation of minimal valuations, we use boxed atoms $\Box^n p$ to compute the minimal valuations, and substitute them into the POS part in $\mathsf{Ant} \to \mathsf{POS}$ and the NEG part in Ant. Therefore, the part of propositonal variables used to compute the minimal valuations are positive in the antecedent, and the part of propositional variables used to receive the minimal valuations are negative in the antecedent Ant and positive in the consequent POS.

In ML(#), however, if we translate formula of the form $\#\varphi \succsim \#\psi$ at world denoted by x, then we need to find an injective function from $R_x \cap [\![\psi]\!]^{\mathbb{M}}$ to $R_x \cap [\![\varphi]\!]^{\mathbb{M}}$. The clause for the standard translation $ST_x(\#\varphi \succsim \#\psi)$ is as follows:

$$\exists S(\forall u\forall v\forall w(Suv \wedge Suw \to v = w) \wedge \forall u\forall v\forall w(Svu \wedge Swu \to v = w)\wedge$$
$$\forall u(\exists v Suv \leftrightarrow Rxu \wedge ST_u(\psi)) \wedge \forall u(\exists v Svu \to Rxu \wedge ST_u(\varphi)))$$

In order to express "the cardinality of a set $\geq$ another set", we need to say that "there is an injective function such that ...", which means that the correspondence language should contain binary pradicate symbol which can be quantified over, therefore a proper correspondence language for ML(#) should be second-order.

In order to compute the first-order correspondence without second-order quantifiers, we also need to eliminate the binary predicate symbol S. So it is not immediate to find a way to eliminate the relevant second-order quantifiers in ML(#). Therefore, we will not make use of the standard translation given here in our correspondence algorithm. However, we can try to combine the results in Section 2 with traditional correspondence theory of modal logic to get a correspondence theory of ML(#), at a price of using infinite conjunctions and disjunctions, which is given in the following sections.

4 Sahlqvist Implications in ML(#)

We use $R_s \in X$ to denote the defining shallow formula φ of the definable subset $X \subseteq \mathbb{N}$, and use $\exists \boldsymbol{p}(R_s \in X)$ to denote $\exists \boldsymbol{p}\varphi$ where all propositional variables in φ are in $\boldsymbol{p}$. We use $\exists \boldsymbol{p}(\#\theta \in X)$ to denote the formula $\exists \boldsymbol{p}\varphi$ stating that the number of θ-successors is in $X \subseteq \mathbb{N}$, and use $\#\theta \in X$ to denote the formula φ. Without loss of generality we assume that for each occurrence of $R_s \in X$ or $\#\theta \in X$, we use a different bunch of auxiliary propositional variables (i.e. the ones in $\boldsymbol{p}$), and these propositional variables are not used anywhere else.

Therefore, to guarantee that $R_s \in X$ and $\#\theta \in X$ are in the right position, we need to make sure that in a Sahlqvist implication $\varphi \to \psi$, $R_s \in X$ and $\#\theta \in X$ occur positively in φ and negatively in ψ.

We define the ML(#)-Sahlqvist implications in the following steps:

- we first define θ as the formulas allowed to occur in $\#\theta \in X$;

$$\theta ::= \Box^n p \mid \bot \mid \top \mid \theta \wedge \theta \mid \theta \vee \theta \mid \Diamond\theta$$

- Then we define the generalized positive (resp. negative) formulas $\mathsf{POS}^\#$ (resp. $\mathsf{NEG}^\#$) as follows, where POS (NEG) denote positive (resp. negative) formulas in the basic modal language:

$$\mathsf{POS}^\# ::= \mathsf{POS} \mid \neg(R_s \in X) \mid \mathsf{POS}^\# \wedge \mathsf{POS}^\# \mid \mathsf{POS}^\# \vee \mathsf{POS}^\# \mid \Box\mathsf{POS}^\#$$

$$\mathsf{NEG}^\# ::= \mathsf{NEG} \mid R_s \in X \mid \mathsf{NEG}^\# \wedge \mathsf{NEG}^\# \mid \mathsf{NEG}^\# \vee \mathsf{NEG}^\# \mid \Diamond\mathsf{NEG}^\#$$

- Then we define the Sahlqvist antecedent $\mathsf{Sahl}^\#$ as follows:

$$\mathsf{Sahl}^\# ::= \Box^n p \mid \bot \mid \top \mid R_s \in X \mid \#\theta \in X \mid \mathsf{NEG} \mid$$
$$\mathsf{Sahl}^\# \wedge \mathsf{Sahl}^\# \mid \mathsf{Sahl}^\# \vee \mathsf{Sahl}^\# \mid \Diamond\mathsf{Sahl}^\#$$

– An ML(#)-Sahlqvist implication is of the form $\mathsf{Sahl}^{\#} \to \mathsf{POS}^{\#}$.

As we can see from the definitions, formulas of the form $\#\varphi \gtrsim \#\psi$ do not occur directly, but are hidden in the abbreviations of the form $R_s \in X$ and $\#\theta \in X$.

5 The Correspondence "Algorithm"

Here we describe the proceduce of obtaining the first-order correspondence (with countable conjunctions and disjunctions) of ML(#)-Sahlqvist implications. Here we use the quotation mark for the word "algorithm" because definable X might be infinite, so the relevant conjunctions and disjunctions might be infinite. The first-order correspondence language for computing the first-order correspondent of ML(#)-Sahlqvist implications contains a binary predicate for the binary relation, individual variables as well as countably infinite conjunction and disjunction.

Given an ML(#)-Sahlqvist implication φ, we first rewrite it as $\forall \boldsymbol{p}\varphi$, where $\boldsymbol{p}$ contains all the propositional variables in φ.

Then we erase all the propositional quantifiers for the auxiliary propositional variables used in subformulas of the form $R_s \in X$ and $\#\theta \in X$ except for those occurring in θ, and rewrite $R_s \in X$ into $\exists \boldsymbol{p}(R_s \in X)$ and $\#\theta \in X$ into $\exists \boldsymbol{q}(\#\theta \in X)$, where $\boldsymbol{p}$ are all propositional variables occurring in $R_s \in X$ and $\boldsymbol{q}$ are all propositional variables occurring in $\#\theta \in X$ and not in θ.

Then we use the following equivalences to replace the left-hand side by the right-hand side in the formula (where $\Diamond_{\geq n}\theta$ is the graded modal formula saying that there are at least n successors satisfying θ):

$$\exists \boldsymbol{q}(\#\theta \in X) \leftrightarrow \bigvee_{n \in X} (\Diamond_{\geq n}\theta \wedge \neg \Diamond_{\geq n+1}\theta)$$

$$\exists \boldsymbol{p}(R_s \in X) \leftrightarrow \bigvee_{n \in X} (\Diamond_{\geq n}\top \wedge \neg \Diamond_{\geq n+1}\top)$$

Then we can apply the standard correspondence theory for graded modal logic (cf. e.g. [2, 4]) to get the first-order correspondence (with countable conjunctions and disjunctions) of the input formula.

References

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