Temporal Logic of Surjective Bounded Morphisms Between Finite Linear Processes

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1 Introduction

In this paper we study a temporal logic for finite linear structures and a surjective bounded morphism between them. We give a modal axiomatization of such structures and also show that any such structure can be uniquely characterised by a temporal formula, up to an isomorphism. As a main theorem we prove Kripke completeness of the proposed axiomatization.

Finite linear structures, *i.e.*, finite sets with a strict linear ordering, naturally arise as representations of a discrete, bounded time flow. Many domains of our everyday practice including time series [2], linear planning [4], [7], scene analysis [1], [9], chain-of-responsibility design pattern in programming [3], [8], *etc.* involve a finite linear structure to represent a sequence of consecutive steps.

In many such scenarios, the processes $(F, <)$ comes with a natural partition $\bigcup_{i\in I} F_i$ of F into *convex* equivalence classes F_i. We call F_i convex if for each $a, b \in F_i$ with $a < b$ and each t with $a < t < b$, we have $t \in F_i$. In such a case, the index set I naturally "inherits" a linear ordering from $(F, <)$.

Mathematically, such a process $(\bigcup_{i\in I} F_i, <)$ can be represented as two temporal linear structures F and I related by means of a bounded morphism.

An example of such a structure comes from the analysis of video data where the linear sequence of image frames is partitioned into intervals (grouped in some way *e.g.* by homogeneous sound moments, or by each interval representing an episode, or a scene). In the area of computer vision, deep learning (DL) methods usually process a video stream as a black box, without looking into the temporal structure or content [6]. By contrast, we aim to represent a high-level knowledge about frames, scenes and their temporal interrelationships and to develop formal languages capable of reasoning about resulting structures [5].

To flesh out this approach a little more, let us consider a conceptual representation of a movie. The raw video data of the movie can simply be represented as a sequence of frames. On a slightly higher level of conceptualization, the same raw data can be understood as a sequence of scenes, where a scene is a subset of logically related consecutive frames. If one also "remembers" which frame belongs to which scene, the following structure emerges:

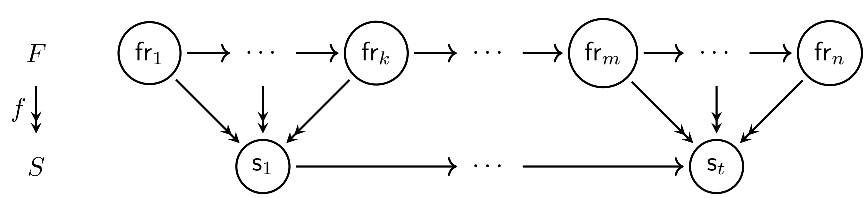

Figure 1: A temporal sequence of the frames F and the scenes S of a movie.

Note that the set of scenes naturally inherits the temporal order from the ordering of the frames.

Our goal in the present paper is to formalise the observed examples as relational structures and study them within the scope of temporal logic.

1.1 Intended structures, syntax and the intended semantics

We represent a temporal sequence of events as a finite strict linear order. To represent a convex partition we use the following definition.

Definition 1.1. Let $(F, <)$ and $(S, <')$ be two strict linear orders. We say that a function $f : F \to S$ is a *bounded morphism* if firstly f is *monotone*, that is $a \leq b$ implies $f(a) \leq f(b)$, and additionally it satisfies the *boundedness* condition: for arbitrary elements $a \in F$ and $b' \in S$, if $f(a) <' b'$ then there exists an element $b \in F$ such that $a < b$ and $f(b) = b'$.

Definition 1.2. A **TES** (Temporal Event Structure) is a tuple $(F, S, <, <', f)$ where $(F, <)$ and $(S, <')$ are finite strict linear orders and $f : F \twoheadrightarrow S$ is an onto bounded morphism.

We use the *functional temporal propositional modal language* $\mathcal{L}$ which consists of formulas ϕ that are built up inductively according to the grammar:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \boxed{\text{-F}}\phi \mid \boxed{\text{F-}}\phi \mid \boxed{\text{-S}}\phi \mid \boxed{\text{S-}}\phi \mid \boxed{\text{f-}}\phi \mid \boxed{\text{-f}}\phi$$

where p ranges over proposition symbols. The logical symbols '$\top$' and '$\bot$', and the dual boolean and modal connectives are defined as usual.

Our intention is to interpret the language $\mathcal{L}$ over an arbitrary **TES** $(F, S, <, <', f)$ in such a way that $\boxed{\text{F-}}$ and $\boxed{\text{-F}}$ range over $(F, <)$; $\boxed{\text{S-}}$ and $\boxed{\text{-S}}$ range over $(S, <')$; and $\boxed{\text{f-}}$ and $\boxed{\text{-f}}$ range over $(F \cup S, f)$, where f is viewed as a (functional) relation.

For example, *truth* of a formula like `ext` $\wedge\ \boxed{\text{F-}}$(`ext`) $\wedge\ \boxed{\text{-F}}$(`int`) at a particular frame is intended to express the property that `"this frame and all the later frames are exterior shots, however all the previous frames are interior shots"`.

The *validity* of a formula like $\boxed{\text{S-}}\,\boxed{\text{S-}}\,\boxed{\text{S-}}\bot$ would mean that the movie consists of at most two scenes. The validity of `Suspense` $\to \langle\text{S-}\rangle$(`Revelation`) would mean that each suspense scene is eventually followed by a revelation scene, *etc.*

2 Axioms and abstract semantics

Let **FS** be a logic defined in the language $\mathcal{L}$ by the following set of axioms and closed under the standard rules of uniform substitution, modus ponens and necessitation.

- All classical tautologies, standard axioms of modal logic **K** for each modal operator;

`Inv:` $p \to \boxed{\text{-x}}\langle\text{x-}\rangle p$ $p \to \boxed{\text{x-}}\langle\text{-x}\rangle p$ $x \in \{F, S, f\}$	`GL:` $\boxed{\text{x-}}(\boxed{\text{x-}}p \to p) \to \boxed{\text{x-}}p$ $\boxed{\text{-x}}(\boxed{\text{-x}}p \to p) \to \boxed{\text{-x}}p$ $x \in \{F, S\}$	`NoBranching:` $\langle\text{-x}\rangle\langle\text{x-}\rangle p \to \langle\text{x-}\rangle p \vee p \vee \langle\text{-x}\rangle p$ $\langle\text{x-}\rangle\langle\text{-x}\rangle p \to \langle\text{x-}\rangle p \vee p \vee \langle\text{-x}\rangle p$ $x \in \{F, S\}$	`Dom-Cod:` $\langle\text{F-}\rangle\top \vee \langle\text{-F}\rangle\top \to \boxed{\text{-f}}\bot$ $\langle\text{S-}\rangle\top \vee \langle\text{-S}\rangle\top \to \boxed{\text{f-}}\bot$
`Surj:` $\langle\text{f-}\rangle\top \vee \langle\text{-f}\rangle\top$	`Bounded:` $\langle\text{f-}\rangle\langle\text{S-}\rangle p \to \langle\text{F-}\rangle\langle\text{f-}\rangle p$	`DomConn:` $\langle\text{f-}\rangle\langle\text{-f}\rangle p \to \langle\text{-F}\rangle p \vee p \vee \langle\text{F-}\rangle p$	
`MinMax:` $\boxed{\text{-f}}\bot \vee \boxed{\text{f-}}\bot$	`Func:` $p \to \boxed{\text{-f}}\,\boxed{\text{f-}}p$	`Monot:` $\langle\text{F-}\rangle\langle\text{f-}\rangle p \to \boxed{\text{f-}}(\langle\text{S-}\rangle p \vee p)$	

Let $\mathrm{Ax}\,(\mathbf{FS})$ denote the axioms of the logic **FS**.

Kripke Semantics for the temporal logic **FS** is provided by Kripke frames with a separate relation for each modal operator and equipped with a valuation function. More precisely a frame $\mathcal{F}$ is a tuple $(W, R_F, R_S, R_f, R'_F, R'_S, R'_f)$ where W is a nonempty set and each of R_F, R_S, R_f, R'_F, $R'_S, R'_f \subseteq W \times W$ is a binary relation on W.

Definition 2.1. Given a frame $\mathcal{F} = (W, R_F, R_S, R_f, R'_F, R'_S, R'_f)$ and a model $\mathcal{M} = (\mathcal{F}, V)$ a truth of a formula at a world $w \in W$ of a model $\mathcal{M}$ is defined in the following way:
For a propositional symbol p define $\mathcal{M}, w \models p$ iff $V(w, p) = 1$;
$\mathcal{M}, w \models \alpha \wedge \beta$ iff $\mathcal{M}, w \models \alpha$ and $\mathcal{M}, w \models \beta$;
$\mathcal{M}, w \models \neg\alpha$ iff it is not the case that $\mathcal{M}, w \models \alpha$;
$\mathcal{M}, w \models \boxed{*}\alpha$ iff for an arbitrary world $v \in W$ with wR_*v we have $\mathcal{M}, v \models \alpha$;
$\mathcal{M}, w \models \boxed{*}\alpha$ iff for an arbitrary world $v \in W$ with wR'_*v we have $\mathcal{M}, v \models \alpha$;
where $* \in \{F, S, f\}$

Definition 2.2. A map f between two relational structures (W, R) and (V, S) is *domain connected* if for any $w, w' \in W$ with $f(w') = f(w)$ we have $w'Rw$, $w' = w$ or wRw'.

Definition 2.3. We will say that a relation $R \subseteq W \times W$ is *trichotomous* if for arbitrary $w, w' \in W$ either wRw' or $w'Rw$. We will say that R is *non-branching*, if for arbitrary elements $a, b, c \in W$, whenever both aRb and aRc hold, then either bRc or cRb holds.

Definition 2.4. We will say that a frame $\mathcal{F} = (W, R_F, R_S, R_f, R'_F, R'_S, R'_f)$ is an **FS**-frame if the following conditions are satisfied: $W = W_F \cup W_S$ where $W_F \neq \emptyset$, $W_S \neq \emptyset$ and $W_F \cap W_S = \emptyset$; $R_F, R'_F \subseteq W_F \times W_F$; $R_S, R'_S \subseteq W_S \times W_S$; R_F, R'_F, R_S and R'_S are non-branching, transitive and well-founded; $R'_F = R_F^{-1}$; $R'_S = R_S^{-1}$; $R_f \subseteq W_F \times W_S$; R_f is a surjective bounded morphism with respect to R_F and R_S; R_f is domain connected; $R'_f = R_f^{-1}$.

The class of **FS**-frames is characterised by the axioms of the temporal logic **FS**.

Theorem 2.1. *For an arbitrary frame $\mathcal{F}$ it holds that $\mathcal{F} \models \mathrm{Ax}\,(\mathbf{FS})$ iff $\mathcal{F}$ is an* **FS**-*frame.*

Clearly a disjoint union of **FS**-frames is again an **FS**-frame. This implies that **FS**-frames can be infinite, and fail the trichotomy property for R_F and R_S, while our intended models, **TES**s are finite with $<$ and $<'$ trichotomous. To retain finiteness and trichotomy, we focus our attention on connected **FS**-frames, *i.e.* on **FS**-frames which cannot be presented as a disjoint union of two **FS**-frames. It turns out that a connected **FS**-frame is in a way isomorphic to a **TES**. We proceed towards establishing this connection.

Definition 2.5. For a given **FS**-frame $\mathcal{F} = (W, R_F, R_S, R_f, R'_F, R'_S, R'_f)$ and elements $u, v \in W$, we will say that there is an *FS-chain* from u to v if there is a finite sequence of points $v_0, v_1, \ldots, v_n$ such that $v_0 = u$, $v_n = v$ and additionally v_i is related to v_{i+1} by an arbitrary relation from the set $\mathrm{Rel} = \{R_F, R_S, R_f, R'_F, R'_S, R'_f\}$.

Definition 2.6. Given an **FS**-frame $\mathcal{F} = (W, R_F, R_S, R_f, R'_F, R'_S, R'_f)$ we will say that $V \subseteq W$ is *connected* in $\mathcal{F}$ if for any two distinct points $u, v \in V$ there is an FS-chain $v_0, v_1, \ldots, v_n$ from u to v such that $v_i \in V$ for each i. We will say that an **FS**-frame $\mathcal{F}$ is connected if its underlying set is connected in $\mathcal{F}$.

Theorem 2.2. *In every connected* **FS**-*frame $\mathcal{F} = (W, R_F, R_S, R_f, R'_F, R'_S, R'_f)$, W is finite; The relation R_F is trichotomous on W_F; The relation R_S is trichotomous on W_S.*

The class of connected **FS**-frames is modally undefinable since it is not closed under disjoint unions, however the next theorem links connected **FS**-frames and **TES**s.

Theorem 2.3. *There is a one-to-one correspondence between the class $\mathcal{C}_{\boldsymbol{TES}}$ of all* ***TES*** *structures and the class $\mathcal{C}_{\mathbf{FS}}$ of all connected* **FS** *-frames.*

Theorem 2.3 allows us to talk about the satisfiability of an $\mathcal{L}$-formula in a model based on a **TES**. Indeed, for a given formula $\phi \in \mathcal{L}$ and a **TES** $\mathcal{F} = (F, S, <, <', f)$ we will say that ϕ is satisfiable at a point $w \in F \cup S$ if there is a valuation V on $F \cup S$ such that $\mathcal{M}, w \models \phi$ where $\mathcal{M} = (\mathcal{F}^*, V)$ is a model based on the **FS**-frame $\mathcal{F}^* = (F \cup S, <, <', f, >, >', f^{-1})$. The notion of validity is also similarly transferred.

The next theorem shows that each **TES** can be fully described by an $\mathcal{L}$-formula, up to an isomorphism.

Theorem 2.4. *Given a* **TES** $\mathcal{F} = (F, S, <, <', f)$*, there is a formula $\phi_{\mathcal{F}} \in \mathcal{L}$ such that for an arbitrary* **TES** $\mathcal{T}$ *we have: $\mathcal{T} \models \phi_{\mathcal{F}}$ iff $\mathcal{T}$ is isomorphic to $\mathcal{F}$.*

Finally, we establish our main finding:

Theorem 2.5. *The logic* **FS** *is sound and complete w.r.t. the class of all* ***TES****s.*

It follows that the logic **FS** has the finite model property and is decidable.

Future Work

It is natural to extend the current work by considering structures with more than two layers *e.g.*, with finitely many layers $(F_1, F_2, \ldots, F_k, f_1, f_2, \ldots, f_{k-1})$ where each layer F_i represents a finite linear order while each $f_i : F_i \to F_{i+1}$ is a surjective bounded morphism. A natural example of such a structure for $k = 3$ would be a set of movie frames, grouped into episodes, these further grouped into scenes, which finally form acts.

The approach and the methods developed in the current study should smoothly generalize to such a setting for any fixed k. The concluding step in this direction would be to axiomatize and study the class of all such structures, for all $k > 1$.

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