Modal Information Logic of Incomparable Fusions

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In [1], van Benthem introduces Modal Information Logic (MIL) to represent a general theory of information. This is done via the semantics of modal logic with a binary modality, where points are interpreted as information states, the relation as an ordering of or inclusion of information, and the modality as describing 'merge' or 'fusion' of information states.

Usually, the relation is taken to be a partial order or preorder, and the binary modality then refers to the corresponding supremum relation (alike, e.g., the conjunction '$\wedge$' in truthmaker semantics). This choice of formalization entails uniqueness of fusions (modulo clusters in the case of preorders). In contrast to this, in this abstract, we explore variants of MIL for formalizing settings in which states can have multiple incomparable fusions. To model this, given a preorder, we consider the induced (quasi-)minimal-upper-bound relation, instead; i.e., we weaken '*least* upper bound' to '*minimal* upper bound'.[1]

This main proof of this talk allows us to deduce three results: (1) perhaps surprisingly, that this change of interpretation of the modality does not result in a different logic: their respective consequence relations coincide; and as corollaries we obtain (2) an axiomatization of the logic(s) and (3) decidability.

Finally, we show that (i) even upon augmenting the logic(s) with an 'informational implication', the corresponding consequence relations remain equal—entailing similar corollaries—and, however, (ii) on finite structures they do come apart.

Putting forth the logics

We begin by providing some pertinent definitions for our logics of concern.

Definition 1 (Language)**.** The language $\mathcal{L}_M$ is defined using a countable set of proposition letters **Prop** and a binary modality '$\langle\min\rangle$'. The formulas $\varphi \in \mathcal{L}_M$ are then given by the following BNF-grammar:

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\min\rangle\varphi\varphi,$$

where $p \in$ **Prop** and $\bot$ is the falsum constant. $\dashv$

Definition 2 (Frames and models)**.** A (Kripke) *preorder frame* is a pair $\mathbb{F} = (S, \leq)$, where S is a set and $\leq$ is a preorder on S (i.e., reflexive and transitive). If $\leq$, furthermore, is anti-symmetric, we call $\mathbb{F}$ a *poset frame*.

A (Kripke) *preorder model* is a triple $\mathbb{M} = (S, \leq, V)$, where $(S, \leq)$ is a preorder frame, and $V :$ **Prop** $\to \mathcal{P}(S)$ is a valuation on S. Moreover, if $(S, \leq)$ is a poset frame, we call $\mathbb{M}$ a *poset model*. $\dashv$

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[1] This change of interpretation is motivated not only by philosophical considerations but also by the completeness proof presented in [3] for the standard MIL of unique fusions. The proof crucially relies on distinguishing between minimal and least upper bounds. Although this axiomatization proof will not be covered in the talk, we will briefly highlight the significance of this distinction in the proof.

Now, before defining the semantics, for the sake of completeness, let us first get clear on what a quasi-minimal upper bound is:

Definition 3. For any preorder $(S, \leq)$ and any $s, t, t' \in S$, we say that s is a *quasi-minimal upper bound* – or simply a *minimal upper bound* – of $\{t, t'\}$ and write $s \in \min\{t, t'\}$:iff

- s is an upper bound of $\{t, t'\}$, i.e., $t \leq s$ and $t' \leq s$; and
- $x \not\leq s$ or $s \leq x$, for all upper bounds x of $\{t, t'\}$.

Note that if $\leq$ is a partial order (i.e., an anti-symmetric preorder), then s is a quasi-minimal upper bound iff it is a minimal upper bound in the usual sense.

In an analogous manner, the notion of quasi-suprema generalizes the standard notion of suprema from posets to preorders. ⊣

Definition 4 (Semantics)**.** For any preorder model $\mathbb{M} = (S, \leq, V)$ and state $s \in S$, *satisfaction* of a formula $\varphi \in \mathcal{L}_M$ at s in $\mathbb{M}$ (written $\mathbb{M}, s \Vdash \varphi$) is defined as follows:

$$
\begin{array}{lll}
\mathbb{M}, s \nVdash \bot, & & \\
\mathbb{M}, s \Vdash p & \textbf{iff} & s \in V(p), \\
\mathbb{M}, s \Vdash \neg\varphi & \textbf{iff} & \mathbb{M}, s \nVdash \varphi, \\
\mathbb{M}, s \Vdash \varphi \vee \psi & \textbf{iff} & \mathbb{M}, s \Vdash \varphi \ \textit{ or } \ \mathbb{M}, s \Vdash \psi, \\
\mathbb{M}, s \Vdash \langle\min\rangle\varphi\psi & \textbf{iff} & \text{there exist } t, t' \in W \text{ such that } \mathbb{M}, t \Vdash \varphi,\ \mathbb{M}, t' \Vdash \psi,\ \text{and } s \in \min\{t, t'\}.
\end{array}
$$

Validity of a formula $\varphi \in \mathcal{L}_M$ in a frame $\mathbb{F}$ (written $\mathbb{F} \Vdash \varphi$) is defined as usual. ⊣

Definition 5 (Logics)**.** The modal information logics of incomparable fusions on preorders and posets, respectively, are defined as

$$MIL^{Min}_{Pre} := \{\varphi \in \mathcal{L}_M \mid (S, \leq) \Vdash \varphi,\ \textit{for all preorder frames } (S, \leq)\},$$

and

$$MIL^{Min}_{Pos} := \{\varphi \in \mathcal{L}_M \mid (S, \leq) \Vdash \varphi,\ \textit{for all poset frames } (S, \leq)\}.$$

Further, omitting the superscripts and writing MIL_{Pre} and MIL_{Pos}, we denote the usual modal information logics (of suprema) on preorders and posets, respectively, in which the modality (typically denoted '$\langle\sup\rangle$' instead of '$\langle\min\rangle$') is interpreted with respect to the induced supremum relation of a preorder/poset $\leq$. ⊣

Results

Having put forth the logics, we are in a place to explain the results achieved. The main proof shows that

$$MIL^{Min}_{Pos} \subseteq MIL_{Pos},$$

which is done through representation. Essentially, this achieves (at least) three results in one go, namely:

1. $MIL^{Min}_{Pre} = MIL^{Min}_{Pos} = MIL_{Pre} = MIL_{Pos}$;
2. An axiomatization of $MIL^{Min}_{Pre} = MIL^{Min}_{Pos}$; and

3. $MIL_{Pre}^{Min} = MIL_{Pos}^{Min}$ is decidable.

This is because [3] axiomatizes MIL_{Pre}, proves it decidable and shows that $MIL_{Pre} = MIL_{Pos}$.

Theorem 6. $MIL_{Pre}^{Min} = MIL_{Pos}^{Min} = MIL_{Pre} = MIL_{Pos}$.

Proof idea. Since (a), clearly,

$$MIL_{Pre}^{Min} \subseteq MIL_{Pos}^{Min},$$

and (b) it is straightforward to show

$$MIL_{Pre} \subseteq MIL_{Pre}^{Min},$$

namely by proving the axiomatization of MIL_{Pre} sound w.r.t. MIL_{Pre}^{Min}, by showing (c)

$$MIL_{Pos}^{Min} \subseteq MIL_{Pos},$$

1., 2. and 3. all follow.

The proof of (c) goes through representation, formalized by the notion of onto p-morphisms. The basic idea is to, given a poset $(S, \leq)$, transform the frame (in a satisfaction-preserving way w.r.t. the supremum interpretation of the modality) so that the induced minimal-upper-bound relation $M_{\leq}$ becomes identical to the induced supremum relation $S_{\leq}$, hence whether the binary modality refers to the supremum or the minimal-upper-bound relation does not matter: the same formulas are satisfied. □

Corollary 7. *The modal information logic of incomparable fusions (on preorders or posets) is decidable and axiomatized by the axiomatization of $MIL_{Pre} = MIL_{Pos}$ from [3].*

As mentioned, the proof of theorem 6 extends to the logics attained by endowing the language with the informational implication '$\backslash$' (suggested in [2]) with semantics

$$\mathbb{M}, t \Vdash \varphi\backslash\psi \qquad \textbf{iff} \qquad \text{for all } t', s \in S, \text{ if } \mathbb{M}, t' \Vdash \varphi \text{ and } s \in \min\{t, t'\}, \text{ then } \mathbb{M}, s \Vdash \psi.$$

To explicate, this gives us:

Theorem 8. $MIL_{\backslash\text{-}Pre}^{Min} = MIL_{\backslash\text{-}Pos}^{Min} = MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$*, where, e.g.,* $MIL_{\backslash\text{-}Pre}^{Min}$ *is the MIL of incomparable fusions on preorders in the language with not only '$\langle\min\rangle$' but also '$\backslash$'.*

Corollary 9. *The modal information logic of incomparable fusions (on preorders or posets) endowed with '$\backslash$' is decidable and axiomatized by the axiomatization of $MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$ from [3].*

Wrapping up, we note the following:

Remark 10. One might conclude that, on preorders and posets, the landscape of MILs is both uniform and decidable:

$$MIL_{Pre} = MIL_{Pos} = MIL_{Pre}^{Min} = MIL_{Pos}^{Min}, \quad MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos} = MIL_{\backslash\text{-}Pre}^{Min} = MIL_{\backslash\text{-}Pos}^{Min}.$$

However, even if the suprema and minima interpretations neither come apart in the basic MIL-setting nor in the $\backslash$-augmented setting, our central proof method of theorem 6 does suggest a setting where they, in fact, do: on *finite* preorders/posets. This is witnessed by the formula

$$(\langle\min\rangle p\top \wedge \langle\min\rangle q\top) \to \langle\min\rangle(\langle\min\rangle pq)\top,$$

which is valid under the minima interpretation, but not under the suprema interpretation. ⊣

References

[1] Johan van Benthem. "Modal logic as a theory of information." In J. Copeland, editor, *Logic and Reality*. 2019.

[2] Johan van Benthem. "Relational Patterns, Partiality, and Set Lifting in Modal Se- mantics." In: Kripke volume in the series 'Outstanding Contributions to Logic'. Ed. by Y. Weiss. Springer. Preprint available at: https://eprints.illc.uva.nl/id/eprint/1773/. Forthcoming.

[3] Søren Brinck Knudstorp. "Modal Information Logics." Master's thesis. Available at: https://eprints.illc.uva.nl/id/eprint/2226/. 2022.