Étale Heyting algebras

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The objective of the present study is an attempt to investigate a class of morphisms of Esakia spaces, called *Esakia local homeomorphisms*, hoping of some evidence that for an Esakia space X the corresponding category LH_{ES}/X of Esakia local homeomorphisms over X enjoys many properties of elementary topoi.

In the paper [5] Andrew M. Pitts proved that any Heyting algebra may occur as the algebra of truth values of a model of the second order propositional calculus. In that paper Pitts also asked whether this result can be generalized to higher orders. One possible reformulation of this question is whether for arbitrary Heyting algebra H there exists an elementary topos with the lattice of all subobjects of its terminal object isomorphic to H.

For Boolean algebras, a positive answer to the Pitts question can be obtained using construction by Peter Freyd [4, Exercise 11 Ch. 9]. Dito Pataraia observed that the elementary topos obtained using the construction of Freyd is equivalent to the category with objects forming certain class LH_{Stone} of local homeomorphisms over the Stone dual space X_B of a Boolean algebra B (unpublished). One thus obtains a topos LH_{Stone}/X_B with the lattice of sub-objects of the terminal object isomorphic to B.

Let us mention that later Pataraia invented an entirely different approach which settled positively the general case of arbitrary Heyting algebras. Unfortunately neither this work is published nor a self-contained text for the complete result does exist yet.

It seems, that it would be useful to also investigate more accessible closely related questions. Even though we believe the approach considered in this talk will not solve the problem of Pitts, it seems that it would be useful to investigate more accessible closely related questions like ours. We try to generalize the Freyd construction from Boolean algebras to Heyting algebras. To generalize the Freyd construction from Boolean algebras to Heyting algebras we use the Esakia duality, which on the level of objects, to a Heyting algebra H assigns an ordered topological space X_H in such a way that this correspondence gives rise to the duality of categories [2], see also [1]. Standard Esakia morphisms $f : X \to Y$ between Esakia spaces X and Y are functions continuous with respect to underlying topologies which are p-morphisms w.r.t the corresponding order relations on X and Y. In what follows ES denotes the category of Esakia spaces and continuous p-morpisms.

Definition 1. *A function between partially ordered sets X and Y is said to be a p-*morphism *if it is order preserving and for any $y \geq f(x)$ with $x \in X$, $y \in Y$ there is an $x' \geq x$ with $f(x') = y$.*

Definition 2. *We call a p-morphism* strict *if such x' is moreover unique. Thus, a strict p-morphism $f : X \to Y$ is a map continuous with respect to the Stone topology and such that for any $y \geq f(x)$ with $x \in X$, $y \in Y$ there is a* unique *$x' \geq x$ with $f(x') = y$.*

In the present talk the role of local homeomorphisms over Esakia spaces play continuous strict p-morphisms. In what follows SE/X denotes the category with objects strict p-morphisms between Esakia spaces with fixed codomain X and morphisms p-morphisms between their domains which make the corresponding triangles commute.

We hope that the categories LH_{ES}/X_H provide useful tools for studying general Heyting algebras, where X_H is the dual Esakia space for a Heyting algebra H. In particular, it seems that finite limits in this category are well behaved and can be explicitly described. On the algebra side this would provide a class of Heyting algebra homomorphisms with manageable amalgamation properties. This might be important in view of the fact that pushouts in the category of Heyting algebras and arbitrary homomorphisms are notoriously difficult to describe.

To explain what we mean, let $\mathcal{H}$ be some variety of Heyting algebras, let $H \in \mathcal{H}$ be an algebra in it, and consider the category $Alg(H)$ of H-algebras.

Objects of $Alg(H)$ are $\mathcal{H}$-homomorphisms $H \to A$, a morphism from $\alpha : H \to A$ to $\alpha' : H \to A'$ being given by a $\mathcal{H}$-homomorphism $\beta : A \to A'$ satisfying $\beta \circ \alpha = \alpha'$. Then, $Alg(H)$ forms the category of all algebras in another variety, which we will also denote by $Alg(H)$: operations of $Alg(H)$ are operations of $\mathcal{H}$ together with constants (nullary operations) c_a, one for each $a \in H$, and identities of the variety $Alg(H)$ are given by identities of $\mathcal{H}$ together with variable-free identities of the form $w(c_{a_1}, \dots, c_{a_n}) = c_{w(a_1,\dots,a_n)}$ for every generating n-ary operation w of $\mathcal{H}$ and every n-tuple $(a_1, ..., a_n)$ of elements of A.

Let finally $\acute{E}t(H)$ be the subvariety of $Alg(H)$ generated by H, i.e. by the identity map of H viewed as the initial object of the category $Alg(H)$. We will call algebras from $\acute{E}t(H)$ *étale H-algebras*, and the $\mathcal{H}$-homomorphisms $H \to A$ that determine étale H-algebras *étale homomorphisms*. In this setting according to the Tarski theorem [6] the category of étale A_0-algebras is the smallest subcategory of $Alg(A_0)$ containing A_0 and closed under products, subalgebras and homomorphic images. This fact strongly suggests that we might benefit from viewing categories of the form $\acute{E}t(A_0)^{op}$ as categories of some sort of local homeomorphisms over the dual object of A_0, formed by gluing together local pieces like one would do in case of topological spaces and open maps [7].

For a Heyting algebra H the corresponding category $\acute{E}t(H)$ of étale H-algebras, i.e. Heyting algebras A equipped with an étale homomorphism $H \to A$, admits a nicer and more explicit description compared to the category of arbitrary H-algebras $Alg(H)$, i.e. those given by arbitrary Heyting algebra homomorphisms $H \to A$. Because of its transparency, we consider a finite case separately. The first part of the subsequent paper is devoted to investigation of the relationship between $\acute{E}t(H)$, the category $Stone^{X_H}$ of presheaves on the dual finite partially ordered set X_H of H with values in the category of Stone spaces, and the variety of H-algebras validating additional axiom (coined by Jibladze in [3]) $(\acute{E}t)_H \quad \bigvee_{h\in H}(x \Leftrightarrow c_h) = \top$, for a finite Heyting algebra H.

The diagram 1 shows connections (functors) between the objects of the study for finite Heyting algebras.

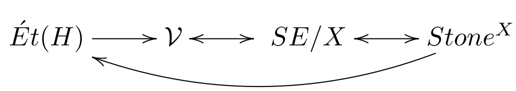

Figure 1: Outline of the finite case

We can summarize the first part of the talk by the following statement:

Theorem 3. *For finite Heyting algebra H, the categories $\acute{E}t(H)$, $\mathcal{V}_{(\acute{E}t)_H}$, SE/X, $Stone^X$ are (dually) equivalent.*

Having settled finite case where the main patterns of a big picture could be traced, in the second part of the talk we attempt to investigate the case of infinite Heyting algebras. Namely we try to establish the the duality between $\acute{E}t(H)$ and ES^X for arbitrary Heyting algebras and their Esakia duals. At the moment of time the last topic is a work in progress. Here we use the techniques of internal category theory [4, Ch. 2] and the key idea for the infinite case is based on the following observation. Strictness of a p-morphism $f : Y \to X$ means that for any $y \in Y$, the map f induces an isomorphism of Esakia spaces $\uparrow y \cong \uparrow f(y)$; translated to the algebra language this means that a H-algebra $h : H \to A$ is étale precisely when for any subdirectly irreducible quotient $q : A \twoheadrightarrow Q$, the composite map qh is surjective.

References

[1] Nick Bezhanishvili. Lattices of intermediate and cylindric modal logics. ILLC Dissertation (DS) Series, Amsterdam, 2006. Institute for Logic, Language and Computation. Thesis, fully internal, Universiteit van Amsterdam.

[2] Leo Esakia. Topological kripke models (russian). *Dokl. Akad. Nauk SSSR*, 214(2):298—-301, 1974. (An English translation appears in Soviet Math Dokladi, 15 (1974), 147-151.).

[3] Mamuka Jibladze. One more notion of relative booleanness. *Proceedings of the Institute of Cybernetics of the Georgian Academy of Sciences*, 2(1-2):52–66, 2002.

[4] Peter Johnstone. *Topos theory.* Academic Press London, New York, 1977.

[5] Andrew M. Pitts. On an interpretation of second order quantification in first order intuitionistic propositional logic. *The Journal of Symbolic Logic*, 57(2):33–52, 1992.

[6] Alfred Tarski. A remark on functionally free algebras. *Annals of Mathematics*, 47(1):163–166, 1946.

[7] Gavin Wraith. Artin glueing. *Journal of Pure and Applied Algebra*, 4(1):345–358, 1974.