Part 3: What to do if a logic does not have Craig Interpolation?

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What to do if a logic does not have Craig interpolation?

Assume L does not have CIP. Two options have been explored:

- What does one have to add to the language of L to restore the CIP? Is there a minimal extension?
- Characterize when φ, ψ have an interpolant in L. How hard is it to decide this? How to compute interpolants if they exist?

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Consider a set $E^+ = \{\varphi_1(a_1), \ldots, \varphi_n(a_n)\}$ of positive examples and a set $E^- = \{\psi_1(b_1), \ldots, \psi_m(b_m)\}$ of negative examples.

For instance, these could be descriptions of drinks or dishes one aims to classify.

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The space of solutions can be reformulated as the set of all interpolants of $\varphi_1(a_1) \vee \dots \vee \varphi_n(a_n)$, $\neg(\psi_1(b_1) \vee \dots \vee \psi_m(b_m))$.

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Plan

- No additional cost of interpolant existence for modal logic of linear orders (K4.3)
- No additional cost of interpolant existence for modal logic of finite strict linear orders (GL.3)
- Approach via formal languages to GL.3
- Minimal temporal languages with CIP
- Exponential additional cost of interpolant existence for: modal logics with nominals, first-order S5, 2-variable fragment, guarded fragment, weak K4.

K4.3 (modal logic of linear orders)

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Let $\Box^+\chi = \chi \wedge \Box\chi$. Consider

$$\varphi = \Diamond(p_1 \wedge \Diamond^+\neg q_1) \wedge \Box(p_2 \to \Box^+ q_1)$$

$\exists q_1.\varphi$ says that p_1 occurs before any occurrence of p_2 (after that anything can happen).

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So $\exists q_1.\varphi \models_{K4.3} \neg\exists q_2.\neg\psi$ and so $\varphi \models_{K4.3} \psi$.

Criterion for Craig interpolant existence (yesterday)

$$\varphi = \Diamond(p_1 \wedge \Diamond^+\neg q_1) \wedge \Box(p_2 \to \Box^+ q_1)$$

$$\neg\psi = \Diamond(p_2 \wedge \Diamond^+\neg q_2) \wedge \Box(p_1 \to \Box^+ q_2).$$

To show that in K4.3 there is no interpolant of φ, ψ we have to find models M_1, x_1 and M_2, x_2 based on linear frames such that for $\Sigma = \{p_1, p_2\}$:

- $M_1, x_1 \models \varphi$;
- $M_2, x_2 \models \neg\psi$;
- that $M_1, x_1 \sim_\Sigma M_2, x_2$.

No interpolant of φ, ψ in K4.3

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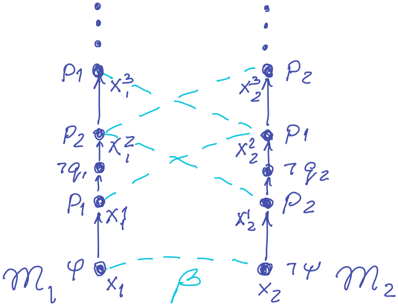

Deciding Interpolant Existence for K4.3

We show the following poly-size bisimilar model property:

Theorem. For any φ, ψ, if φ and ψ are satisfiable in Σ-bisimilar models based on linear frames, then they are satisfiable in poly-size Σ-bisimilar models based on linear frames.

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Corollary. Interpolant existence is in coNP for K4.3.

Descriptive frames

A general frame $F = (W, R, P)$ consists of a frame (W, R) and a set of internal sets $P \subseteq 2^W$ closed under the Booleans and the operator

$$\Diamond^F X = \{x \in W \mid \exists y \in X \, xRy\}.$$

$F = (W, R, P)$ is called descriptive if the following conditions hold for any $x, y \in W$ and any $\mathcal{X} \subseteq P$:

(dif) $x = y$ iff $\forall X \in P \, (x \in X \leftrightarrow y \in X)$,

(ref) xRy iff $\forall X \in P \, (y \in X \to x \in \Diamond^F X)$,

(com) if $\mathcal{X} \subseteq P$ has the finite intersection property, that is, $\bigcap \mathcal{X}' \neq \emptyset$ for every finite $\mathcal{X}' \subseteq \mathcal{X}$—then $\bigcap \mathcal{X} \neq \emptyset$.

Interpolant Existence based on Descriptive Frames

A d-frame based model $M = (W, R, P, V)$ consists of a descriptive frame (W, R, P) and a model (W, R, V) with $V(p_i) \in P$ for all p_i.

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Theorem [Completeness] For every normal modal logic L, $\models_L$ is determined by d-frame based models with underpinning descriptive frames validating L.

Theorem. The following conditions are equivalent for any normal modal logic L, formulas φ, ψ and $\Sigma = \text{sig}(\varphi) \cap \text{sig}(\psi)$:

- there does not exist an interpolant for φ, ψ in L
- φ and $\neg\psi$ are satisfiable in Σ-bisimilar *d*-frame based models with descriptive frames validating L.

Back to poly-size bisimilar models for K4.3

Assume $M_1 = (W_1, R_1, P_1, V_1)$, $M_2 = (W_2, R_2, P_2, V_2)$ and

$$M_1, w_1 \models \varphi_1, \quad M_2, w_2 \models \varphi_2$$

such that $M_1, w_1 \sim_\Sigma M_2, w_2$.

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(1) For $i = 1, 2$, take w_i and for $\chi \in \text{sub}(\varphi_i)$ a maximal point in W_i satisfying χ (exist as we have descriptive frames). Let V_i be the resulting sets.

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(1) For $i = 1, 2$, take w_i and for $\chi \in \mathrm{sub}(\varphi_i)$ a maximal point in W_i satisfying χ (exist as we have descriptive frames). Let V_i be the resulting sets.

(2) Take for $w \in V_1 \cup V_2$ a maximal point in W_i satisfying the same full Σ-type as w (all Σ-formulas true in w). Exist as we have descriptive frames and by Σ-bisimilarity. The induced models and Σ-bisimulation are as required.

The modal logic of strict finite orders (GL.3)

Theorem [Kurucz, W, Zakharyaschev] Craig interpolant existence is coNP-complete for all finite axiomatizable normal extensions of K4.3.

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We consider GL.3, the modal logic of strict finite orders axiomatized by adding to K4.3 the Gödel-Löb axiom

$$\Box(\Box p \rightarrow p) \rightarrow \Box p.$$

It is valid in a transitive frame (W, R) iff the frame does not contain and infinite ascending R-chain $w_0 R w_1 R \cdots$.

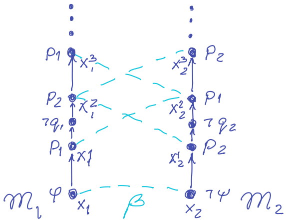
GL.3 (Logic of finite strict linear orders)

$$\varphi = \Diamond(p_1 \wedge \Diamond^+\neg q_1) \wedge \Box(p_2 \rightarrow \Box^+ q_1)$$

$\exists q_1.\varphi$ says that p_1 occurs before any occurrence of p_2

$$\neg\psi \equiv \Diamond(p_2 \wedge \Diamond^+\neg q_2) \wedge \Box(p_1 \rightarrow \Box^+ q_2)$$

φ and $\neg\psi$ can't be satisfied $\{p_1, p_2\}$-bisimilar finite strict orders.



Descriptive frames to the rescue

Consider $F_k = (W_k, R_k, P_k)$ with (W_k, R_k) depicted below and P_k the boolean closure of singletons $\{n\}$ and

$$X_i = \{a_i\} \cup \{kn + i \mid n < \omega\}$$

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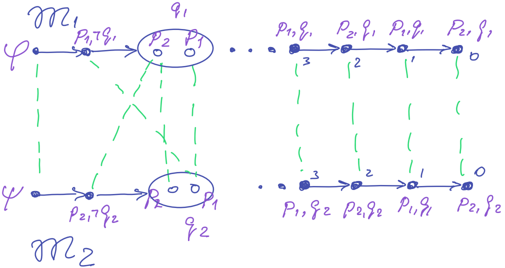
Observation Finite sequences of such frames and irreflexive nodes validate GL.3. Call them basic GL.3-frames.

$\{p_1, p_2\}$-bisimilar basic GL.3 frames

φ and $\neg\psi$ satisfied in $\{p_1, p_2\}$-bisimilar basic GL.3 frames:



coNP Upper Bound for Interpolant Existence in GL.3

Theorem For any φ, ψ,

if φ and ψ are satisfiable in Σ-bisimilar descriptive frames validating GL.3,

then they are satisfiable in Σ-bisimilar basic GL.3 frames with only polynomially many components.

Different approach for LTL using algebraic techniques

Theorem [Henckell 1988, Place, Zeitoun 2016] For any disjoint regular languages (of finite words), R_1, R_2, it is decidable (in ExpTime) whether there exists an FO-definable language L separating them:

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As regular languages are models of $\exists \mathbf{q}.\varphi_1$ and $\forall \mathbf{q}.\varphi_2$ with φ_1, φ_2 in LTL (equivalently FO), this result states that interpolant existence for LTL over strict finite orders is decidable.

Minimal Language Extension of GL.3 with CIP

Let MSO denote monadic second-order logic over structures $F = (W, R, p_1^F, \ldots)$ with $p_1, \ldots$ unary relation symbols corresponding to propositional atoms.

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MSO is the smallest extension of ML over finite strict linear orders with CIP.
Equivalently, the extension of ML with an operator for “next” and the fixpoint operator μ is the smallest extension of ML with CIP over strict finite orders.

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Note. The notion of an "extension" has to be defined. An important condition is closure on substitutions: roughly, if $\varphi(p) \in \mathcal{L}$ and $\psi \in \mathcal{L}$, then $\varphi(\psi) \in \mathcal{L}$. Closure under negation is also used.

Results for K4.3 not typical

The following are logics where interpolant existence is approximately one exponential harder than entailment:

- Guarded fragment and two-variable fragment [Jung and W 2021];
- Modal logics with nominals [Artale et al. 2021];
- One-variable fragment of first-order S5 [Kurucz, W, Zakharyschev];
- $wK4 = K \oplus \Diamond\Diamond p \rightarrow (p \vee \Diamond p)$, the logic of the derivative operator [not yet published].

One can satisfy φ, ψ in Σ-bisimilar models only if they have at least exponentially many Σ-bisimilar nodes.

Illustration: Modal logic with nominals

We add to modal logic a countably infinite set of nominals (denoted a, b, and so on), propositional atoms that have to be interpreted as singletons. For simplicity we also add universal role $\Box_u$.

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Theorem. Interpolant existence is 2ExpTime-complete.

Why is Σ-bisimilarity hard?

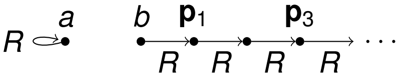
$\mathbf{p} = p_0, \ldots, p_{n-1} \notin \Sigma$ used to encode counter up to $2^n - 1$ with

$\mathbf{p}_i$ short for 'the number encoded by $\mathbf{p}$ is i'.

Let

- $\varphi = a \wedge \Diamond a$
- $\psi = \mathbf{p}_0 \wedge \bigwedge(\Box_u(\mathbf{p}_i \rightarrow \Box \mathbf{p}_{i+1})$

Σ-bisimilar models of φ and ψ:



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Now, as we have to coordinate what happens on the R-chain, work with

sets T of types satisfiable in Σ-bisimilar nodes

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Double exponentially many sets of sets of types 2EXPTIME

wK4 $= \text{K4} \oplus \Diamond\Diamond p \to (p \vee \Diamond p)$

The logic of the derivative operator on topological spaces, introduced by Esakia (based on Tarski/McKinsey):

- $d(X)$ is the set of all points x such that every neighbourhood of x contains a point $y \in X$ with $y \neq x$.

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Frames for wK4 satisfy

$$xRyRz \Rightarrow x = z \vee xRz,$$

so are partial-orders of clusters of possibly irreflexive nodes.

wK4 does not have CIP

Consider

$$\varphi = \Diamond\Diamond p \wedge \neg\Diamond p$$

Then $M, w \models \exists p.\varphi$ iff $M \models \exists y(wRyRw \wedge \neg(wRw))$

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$$\psi = q \rightarrow \Diamond\Diamond q$$

Then $M, w \models \forall q.\psi$ iff $M \models \exists y wRyRw$.

Hence wK4$\models \varphi \rightarrow \psi$.

But there is no interpolant.