Gödel Logics: On the Elimination of the Absoluteness Operator

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Gödel logics G_V, (where set of truth values V is a closed subset of $[0,1]$ containing 0 and 1) form an essential class of intermediate logics, those that are stronger than intuitionistic logic yet weaker than classical logic. The language is standard (propositional, first-order) with countably infinite propositional variables A_i, connectives $\wedge$, $\vee$, $\rightarrow$, and the constants $\bot$ for "false" and $\top$ for "true'; Atomic formulas include propositional variables, and truth constants.

Definition 1. *A valuation $\mathcal{I}$ based on V is a function from the set of propositional variables into V given as follows:*

(1) $\mathcal{I}(\bot) = 0$, (2) $\mathcal{I}(\top) = 1$, (3) $\mathcal{I}(A \wedge B) = \min\{\mathcal{I}(A), \mathcal{I}(B)\}$,
(4) $\mathcal{I}(A \vee B) = \max\{\mathcal{I}(A), \mathcal{I}(B)\}$, (5) $\mathcal{I}(\forall x A(x)) = \mathit{inf}\{\mathcal{I}(A(u))\, u \in U_{\mathcal{I}}\}$,
(6) $\mathcal{I}(\exists x A(x)) = \mathit{sup}\{\mathcal{I}(A(u))\, u \in U_{\mathcal{I}}\}$, (7) $\mathcal{I}(A \supset B) = \begin{cases} \mathcal{I}(B), & \mathcal{I}(A) > \mathcal{I}(B), \\ 1, & \mathcal{I}(A) \leqslant \mathcal{I}(B). \end{cases}$

A formula in Gödel logic is valid *iff the formula evaluates to 1 under every interpretation. The Gödel logic G_V is defined as the set of valid formulas. Note that the validity and 1-satisfiability are not dual in Gödel logic.*

The asymmetry between the truth values 0 and 1 in Gödel logics, stemming from continuity conditions at 1, motivates the introduction of the absoluteness operator Δ [1], which precisely identifies formulas evaluating to 1

$$\mathcal{I}(\triangle A) = \begin{cases} 1 & \text{if } \mathcal{I}(A) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1. *There is no connective $\triangle$ definable with other connectives and variables*

Proof. There are a finite number of 1-variable functions in Gödel logic.

$$\top, \bot, A, \neg A, \neg A \vee A, \neg A \supset A$$

Assume that $\triangle$ is definable by some of the function F, i.e., $\triangle(A) \leftrightarrow F(A)$. Now we look at the F in G_3, because if $\triangle$ is not definable in G_3 then it is not definable in all larger Propositional Gödel logics.
The following truth table shows that none of them defines $\triangle$ and they are closed under composition by all connectives:

A	$\neg A$	$\top$	$\bot$	$A \vee \neg A$	$\neg A \rightarrow A$	$\triangle A$
0	1	1	0	1	0	0
1/2	1/2	1	0	1/2	1	0
1	0	1	0	1	1	1

Therefore we introduce the connective $\triangle$ extending the language. □

Existing literature [1] establishes that $\triangle$ is generally non-eliminable, aligning closely with modal logic S4 augmented by tertium non datur. Here, we demonstrate that $\triangle$ can be entirely eliminated under a novel *restricted semantics* characterized by interpreting all propositional atoms, except the logical constant for truth ($\top$), strictly below 1. To indicate the use of such semantics, we denote Gödel logics by G_V^-. The valuation remains the same as defined above.

Proposition 2. *The formula $F(x_1 \dots x_n)$ in Gödel logic $G_\triangle^-$ is valid in the restricted semantics iff $(\neg\triangle x_1 \wedge \dots \wedge \neg\triangle x_n) \to F$ is valid in Gödel logics $G_\triangle$ with $\triangle$ in the usual semantics .*

We achieve this elimination by systematically transforming formulas containing $\triangle$ into chain normal forms, decomposing complex expressions into linear chains devoid of the absoluteness operator.

Definition 2. *A* chain *C over the set of propositional variables $x = [x_1, ...x_n]$ is an expression*

$$\bot \rhd_1 X_{\Pi(1)} \rhd_2 \top \rhd_1 X_{\Pi(2)} \rhd_3 \cdots \rhd_n X_{\Pi(n)} \rhd_{n+1} \top$$

where Π is permutation on $[1 \dots n]$ and $\rhd_i \in \{<, \leftrightarrow\}$. We denote a chain in the restricted semantics by C^-. A chain C^- does not provide the equivalence of an form $a \leftrightarrow \top$ for any variable a.

Proposition 3. *In standard semantics, the full disjunction of chains $\bigvee C$ is valid in all Gödel logics. Similarly, the disjunction of chains in the restricted semantics $\bigvee C^-$ is valid in all Gödel logics under the restricted semantics.*

Definition 3. *The* chain normal form without $\triangle$ *for a formula with $\triangle$ in the restricted semantics is obtained from the expression $\bigvee C^- \wedge \psi_c(a)$ where $\bigvee C^-$ are all chains without $\triangle$ in the restricted semantics and $\psi_c(a)$ is an evaluation over the chain C^- of a with variables among the variables of the chain, after the following steps:*
1) If the evaluation of a is false we delete the chain,
2) If the evaluation of a is true, we leave the chain as it is,
3) if the evaluation of a is an atom and is not 1 we delete the whole chain.

Propositional Gödel logics can be identified through well-founded linear Kripke structures. The $\triangle$-operator in Gödel logics can be interpreted as a stability operator, meaning: $\triangle A$ holds if and only if A is true in all future and past worlds. In restricted semantics it means that all atoms besides $\top$ are assigned 0 in the "downmost world".

It is important to note that the so-called equivalence principle $A \leftrightarrow B \Rightarrow E(A) \leftrightarrow E(B)$ for a given context E generally hold for Gödel logics without $\triangle$ even in first-order language because of the full deduction theorem. But it does not hold when $\triangle$ is presented also for the restricted semantics.

Example 1. *The specific case $A \leftrightarrow B \Rightarrow \triangle(A) \leftrightarrow \triangle(B)$ for the $\triangle$ operator also fails in the restricted semantics. To illustrate, let assign to A value 1 and to B some value strictly between 0 and 1. In this case, $\triangle A$ is 1 and $\triangle B$ is 0, yet $A \leftrightarrow B$ is not 0. This contradiction demonstrates why the principle does not hold universally.*

Consequently, we must modify the evaluation process for $\triangle$ to accommodate this limitation. First, we show that

Lemma 1. *For any formulas A, B, C, D, and any context function E, the following implication holds: from $A \vee B \wedge (C \leftrightarrow D)$ we derive $A \vee B \wedge (E(C) \leftrightarrow E(D))$*

This lemma holds for both standard and restricted semantics. Moreover, in standard Gödel logics without $\triangle$, it ensures full equivalence, as these logics satisfy the full deduction theorem.

Lemma 2. *The transitive closure of equivalence in the restricted semantics holds, i.e., for any formulas C, D, L:*

$$(A \vee (B \wedge (C \leftrightarrow D))) \wedge (D \leftrightarrow L) \Rightarrow (A \vee B \wedge (C \leftrightarrow L)).$$

Lemma 3. *Given any expression $\triangle(C \vee (D \wedge a))$ for some variable a in the restricted semantics where C, D are valid expressions we can eliminate $\triangle$ obtaining C*

$$\frac{\triangle(C \vee (D \wedge a))}{C}$$

Theorem 1. *In the restricted semantics each formula F with $\triangle$ is equivalent to a disjunction of chains without $\triangle$.*

Proof. We proceed by expressing the given formula F in terms of its chain decomposition. Consider the disjunction of chains $C_1 \vee \cdots \vee C_n$ corresponding to the variables in F under the assumption that no atoms besides $\top$ are interpreted at 1. Distributing F over this disjunction yields $C_1 \wedge F \vee \cdots \vee C_n \wedge F$. Since each chain evaluates to 1 or 0 based on the fact that $\triangle A$ is 0 for a variable A. The elimination process follows from the properties of chain decomposition and validity preservation in the restricted semantics. □

By reformulating formulas into chain normal forms, we ensure that $\triangle$ can be systematically removed while preserving the validity of equivalence. The final form is a disjunction of chains without $\triangle$, which evaluates to 1. We illustrate elimination method through explicit example:

Example 2. *Given a simple formula $F := a \vee \triangle(a \vee a \to \bot)$, the corresponding chain decomposition yields three chains in standard semantics:*

$$(\bot \leftrightarrow a) < \top, (\bot < a) < \top, (\bot < 1 \leftrightarrow a)$$

for some variable a. Note that by definition the last chain is not valid in restricted semantics. Therefore, we have the following disjunction of chains in restricted semantics $(\bot \leftrightarrow a) < \top \vee (\bot < a) < \top$. Now we construct the chain normal form in the restricted semantics $(\bot \leftrightarrow a) < \top \wedge F \vee (\bot < a) < \top \wedge F$. Note that we evaluate from inner most first.

Evaluation of the first chain:	***Evaluation of the second chain:***
$a \vee \triangle(a \vee \top)$	$a \vee \triangle(a \vee \bot)$
$a \vee \triangle\top$	$a \vee \triangle a$
$a \vee \top$	$a \vee \bot$
$\top$	a
	$\bot$

and we get

$$a \vee \triangle(a \vee a \to \bot) \leftrightarrow \neg a.$$

This example also illustrates the fact that the restricted semantics is not closed under substitution. Assume we substitute $\top$ for a, we obtain $\top \vee \triangle(\top \vee \top \to \bot) \leftrightarrow \neg\top$ and consequently $\top \leftrightarrow \bot$.

The argument used for the propositional case does not extend to the first-order case. For example, when 1 is not isolated and does not belong to a perfect set, however 0 is isolated or does belong to a perfect set, the first-order Gödel logic with $\triangle$ is not recursively enumerable, while the first-order logic without $\triangle$ is. This holds both for standard and restricted semantics. Therefore there is not even an effective validity equivalence elimination of $\triangle$, and obviously no valid equivalence as in the propositional case.

Motivation for eliminating $\triangle$ is multifaceted. Primarily, it simplifies the study of prenex fragments, facilitating clearer semantic interpretations, quantifier manipulations, and decision procedures. Additionally, elimination clarifies precisely when $\triangle$ affects logical validity of a formula, and it reveals how $\triangle$ influences logic completeness, especially at the first-order level, especially in contexts where the truth value 1 is neither isolated nor part of a perfect set. This insight directly informs complexity-theoretic classifications of first-order Gödel logics and contributes to identifying completeness conditions.

Our introduction of restricted semantics, yields notable consequences, including axiomatization without $\triangle$, recursive inseparability of certain first-order sentences under these semantics, and restoration of the unlimited deduction theorem, typically restricted in standard Gödel logics incorporating $\triangle$. Furthermore, this framework suggests broader applications, inviting investigation into similar semantic restrictions in related intermediate and modal logics, potentially influencing logical properties analogous to those found in S4-like structures.

References

[1] Matthias Baaz. Infinite-valued Gödel logic with 0-1-projections and relativisations. In Petr Hájek, editor, *Gödel'96: Logical Foundations of Mathematics, Computer Science, and Physics*, volume 6 of *Lecture Notes in Logic*, pages 23–33. Springer-Verlag, Brno, 1996.

[2] Matthias Baaz and Norbert Preining. Quantifier elimination for quantified propositional logics on Kripke frames of type omega. *Journal of Logic and Computation*, 18:649–668, 2008.

[3] Matthias Baaz, Norbert Preining, Gödel–Dummett logics, in: Petr Cintula, Petr Hájek, Carles Noguera (Eds.) *Handbook of Mathematical Fuzzy Logic* vol.2, College Publications, 2011, pp.585–626, chapterVII.

[4] Matthias Baaz and Norbert Preining. On the classification of first order Goedel logics. *Ann. Pure Appl. Log* 170:36–57, 2019.

[5] Matthias Baaz, Norbert Preining, and Richard Zach. Completeness of a hypersequent calculus for some first-order Göde logics with delta. In *36th International Symposium on Multiple-valued Logic (ISMVL 2006)*. IEEE Computer Society, 2006.

[6] Arnold Beckmann and Norbert Preining. Linear Kripke frames and Gödel logics. *Journal of Symbolic Logic*, 72(1):26–44, 2007.

[7] Michael Dummett. A propositional calculus with denumerable matrix. *Journal of Symbolic Logic*, 24(2): 97–106, 1959.

[8] Kurt Gödel. Zum intuitionistischen Aussagenkalkül. *Anzeiger Akademie der Wissenschaften Wien*, 69: 65–66, 1932.

[9] Hiroakira Ono. Kripke models and intermediate logics. *Publications of the Research Institute for Mathematical Sciences Kyoto University*, 6:461–476, 1971.